Reverse Cholesky Factorizaton

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Background and Motivation

Given $P \in B(\ell^2(\mathbb{N}))$ positive semidefinite, the LU-decomposition, a.k.a., the Cholesky algorithm, yields $P = LL^*$ with L lower triangular.

What about $P = UU^*$ with U upper triangular? Why bother?

Answer: UU^* factorizations related to problems in complex analysis.

Szego's Theorem: If $p \geq 0$ on circle, then $p = |f|^2, a.e.$ with f analytic on disk iff $ln(p)$ integrable.

From this it follows that a Toeplitz operator with symbol p, T_p factors as $T_p = UU^*$ iff $ln(p)$ integrable.

Many multivariable analogues still open.

Multivariable Case

o.n.b. for $\ell^2(\mathbb{Z}^+)^{\otimes M}$ given by $e_I = e_{i_1} \otimes \cdots \otimes e_{i_M}$ where $I = (i_1, ..., i_M)$. If $P \in B(\ell^2(\mathbb{Z}^+)^{\otimes M})$ write $P = (p_{I,J})$ where $p_{I,J} = \langle Pe_J, e_I \rangle$. $U = (u_{I,J})$ is **M-upper triangular** iff $u_{I,J} = 0$ whenever $\exists K,\ i_K \le j_K$(equivalently, $U \in Alg(\mathbb{Z}^+)^{\otimes M}$–the tensor product of nest algebras).
Set $z^I = z_1^{i_1} \cdots z_M^{i_M}$.

AFMP: If $P = (p_{I,J}) \in B(\ell^2(\mathbb{Z}^+)^{\otimes M})^+$, then $K_P(z, w) = \sum_{I,J} p_{I,J} z^I \bar{w}^J$ is the reproducing kernel for a space of analytic functions on $\mathbb{D}^M$, denoted $\mathcal{H}(K_P)$.
$P = UU^*$ with U M-upper iff the polynomials are dense in $\mathcal{H}(K_P)$.

Results

PW: Let $P \in B(\ell^2(\mathbb{Z}^+)^{\otimes M})^+$. If for each J, $p_{I,J}$ is non-zero for only finitely many I, then $P = UU^*$ with U M-upper.
Weak Fejer-Reisz: Let p be a positive trig polynomial in M variables, then $T_p = UU^*$ with U M-upper.

Proof: Set $\phi_J(z) = (J!) \sum_I p_{I,J} z^I$(a polynomial), we prove that $\phi_J \in \mathcal{H}(K_P)$ and for $f \in \mathcal{H}(K_P)$,

$$f^{(J)}(0) = \langle f, \phi_J \rangle_{\mathcal{H}(K_P)}.$$

Hence, $f \perp \{polynomials\} \implies f^{(J)}(0) = 0, \forall J \implies f = 0$

The following extends results of Anoussis-Katsoulis in *Factorisation in nest algebras I, II*:

PW: Let $P \in B(\ell^2(\mathbb{Z}^+)^{\otimes M})^+$. The following are equivalent:

1. $P = UU^*$ for some $U \in Alg(\mathbb{Z}^+)^{\otimes M}$,
2. $Ran(P^{1/2}) = Ran(C)$ for some $C \in Alg(\mathbb{Z}^+)^{\otimes M}$

In particular, if P is invertible, then P factors.

$1 \implies Ran(P^{1/2}) = Ran(U)$ by Douglas factorization.
For $2 \implies 1$, AFMP showed that
$f = \sum_I a_I z^I \in \mathcal{H}(K_P) \iff \sum_I a_I e_I \in Ran(P^{1/2})$.
Now show "polynomials" are dense in $Ran(C)$.

Both our results(and proofs) hold for operator-valued matrices and vector-valued reproducing kernel Hilbert spaces as well.

Thanks!