Reverse Cholesky Factorization

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Given $P \in B(\ell^2(\mathbb{N}))$ positive semidefinite, the LU-decomposition, a.k.a., the Cholesky algorithm, yields $P = LL^*$ with $L$ lower triangular.

What about $P = UU^*$ with $U$ upper triangular? Why bother?

Answer: $UU^*$ factorizations related to problems in complex analysis.

**Szegö's Theorem:** If $p \geq 0$ on circle, then $p = |f|^2$, a.e. with $f$ analytic on disk iff $\ln(p)$ integrable.

From this it follows that a Toeplitz operator with symbol $p$, $T_p$ factors as $T_p = UU^*$ iff $\ln(p)$ integrable.

Many multivariable analogues still open.
o.n.b. for $\ell^2(\mathbb{Z}^+)^{\otimes M}$ given by $e_I = e_{i_1} \otimes \cdots \otimes e_{i_M}$ where $I = (i_1, \ldots, i_M)$. If $P \in B(\ell^2(\mathbb{Z}^+)^{\otimes M})$ write $P = (p_{I,J})$ where $p_{I,J} = \langle Pe_J, e_I \rangle$. $U = (u_{I,J})$ is M-upper triangular iff $u_{I,J} = 0$ whenever $\exists K, i_K \leq j_K$ (equivalently, $U \in Alg(\mathbb{Z}^+)^{\otimes M}$—the tensor product of nest algebras).

Set $z^I = z^i_1 \cdots z^{i_M}_M$.

**AFMP:** If $P = (p_{I,J}) \in B(\ell^2(\mathbb{Z}^+)^{\otimes M})^+$, then $K_P(z, w) = \sum_{I,J} p_{I,J} z^I \bar{w}^J$ is the reproducing kernel for a space of analytic functions on $\mathbb{D}^M$, denoted $\mathcal{H}(K_P)$. $P = UU^*$ with $U$ M-upper iff the polynomials are dense in $\mathcal{H}(K_P)$. 
**PW:** Let \( P \in B(\ell^2(\mathbb{Z}^+) \otimes^M)^+ \). If for each \( J \), \( p_{I,J} \) is non-zero for only finitely many \( I \), then \( P = UU^* \) with \( U \) \( M \)-upper.

**Weak Fejer-Reisz:** Let \( p \) be a positive trig polynomial in \( M \) variables, then \( T_p = UU^* \) with \( U \) \( M \)-upper.

**Proof:** Set \( \phi_J(z) = (J!) \sum_I p_{I,J} z^I \) (a polynomial), we prove that \( \phi_J \in \mathcal{H}(K_P) \) and for \( f \in \mathcal{H}(K_P) \),

\[
f^{(J)}(0) = \langle f, \phi_J \rangle_{\mathcal{H}(K_P)}.
\]

Hence, \( f \perp \{\text{polynomials}\} \implies f^{(J)}(0) = 0, \forall J \implies f = 0 \)
The following extends results of Anoussis-Katsoulis in *Factorisation in nest algebras I, II*:

**PW:** Let $P \in B(\ell^2(\mathbb{Z}^+) \otimes M)^+$. The following are equivalent:

1. $P = UU^*$ for some $U \in \text{Alg}(\mathbb{Z}^+) \otimes M$,
2. $\text{Ran}(P^{1/2}) = \text{Ran}(C)$ for some $C \in \text{Alg}(\mathbb{Z}^+) \otimes M$

In particular, if $P$ is invertible, then $P$ factors.

1 $\implies$ $\text{Ran}(P^{1/2}) = \text{Ran}(U)$ by Douglas factorization.

For 2 $\implies$ 1, AFMP showed that

$$f = \sum_I a_I z_I \in \mathcal{H}(K_P) \iff \sum_I a_I e_I \in \text{Ran}(P^{1/2}).$$

Now show “polynomials” are dense in $\text{Ran}(C)$.
Both our results (and proofs) hold for operator-valued matrices and vector-valued reproducing kernel Hilbert spaces as well.

Thanks!