CO 602/CM 740: Fundamentals of Optimization
Fall 2017
Problem Set 1
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Due: 2017-Sep-21 in lecture.

1. Consider the model presented in lecture to develop a predictive formula for a score g_i of student i, $i = 1, \ldots, m$, from historical data $S(i,j)$, $j = 1, \ldots, n$. Recall that finding the model coefficients $x_1, \ldots, x_{n+1}$ entails minimizing $f(\mathbf{x})$ where

$$f(\mathbf{x}) = \max_{i=1,\ldots,m} \left| \sum_{j=1}^{n} S(i,j)x_j + x_{n+1} - g_i \right|.$$

Suppose in addition that we wish to include constraints that $|x_j| \le r$ for each $j = 1, \ldots, n+1$, where r is a given positive number. (Such a constraint is said to "regularize" the problem; it prevents the occurrence of large coefficients. Large coefficients can make the model less robust for future unseen data.) Show that minimizing $f(\mathbf{x})$ subject to these additional constraints is still an instance of linear programming.

2. A *line* in $\mathbf{R}^n$ is defined to be a set of the form $L = \{\mathbf{v}_0 + \alpha\mathbf{v}_1 : \alpha \in \mathbf{R}\}$ where $\mathbf{v}_0, \mathbf{v}_1$ are given vectors in $\mathbf{R}^n$ with $\mathbf{v}_1 \ne \mathbf{0}$.

(a) Can the feasible region for an LP that is in standard equality form (s.e.f.) contain a line? If yes, give an example; if no, prove that an s.e.f. feasible region can never contain a line.

(b) Let $A \in \mathbf{R}^{m\times n}$ and $\mathbf{b} \in \mathbf{R}^m$ define the polyhedron $P = \{\mathbf{x} \in \mathbf{R}^n : A\mathbf{x} \ge \mathbf{b}\}$. Show that if $\mathrm{rank}(A) = n$, then P cannot contain a line. Conversely, show that if P is nonempty and $\mathrm{rank}(A) < n$, then P must contain a line.

3. Suppose matrix $A \in \mathbf{R}^{n\times n}$ is nonsingular. Suppose $\mathbf{x} \in \mathbf{R}^n$ is a nonzero vector that is orthogonal to columns $1, \ldots, n-1$ of A. Finally, suppose $\mathbf{d}$ is a vector such that $\mathbf{d}^T\mathbf{x} \ne 0$. Show that $[A(:, 1 : n-1), \mathbf{d}]$ is a nonsingular matrix.

[Hint: suppose $[A(:, 1 : n-1), \mathbf{d}]\mathbf{y} = 0$. From this, conclude $\mathbf{y} = \mathbf{0}$. At one step of the argument, take an inner product of both sides with $\mathbf{x}$.]

4. An *affine set* $W \subset \mathbf{R}^n$ is the set of solutions to a system of linear equations: $W = \{\mathbf{x} \in \mathbf{R}^n : A\mathbf{x} = \mathbf{b}\}$ where A is a given $m \times n$ matrix for some m and $\mathbf{b}$ is a given m-vector.

(a) Show based on the definition of convexity that an affine set is convex.

(b) Suppose that $W = \{\mathbf{x} \in \mathbf{R}^n : A\mathbf{x} = \mathbf{b}\}$ is a nonempty affine set. Let $T_1 = \{\mathbf{s} : A\mathbf{s} = \mathbf{0}\}$ and $T_2 = \{\mathbf{x}_1 - \mathbf{x}_2 : \mathbf{x}_1, \mathbf{x}_2 \in W\}$. Prove that $T_1 = T_2$. Note: this set (either definition) is called the *tangent space* of W, denoted $\mathcal{T}W$.

(c) Show that lines (as defined in Q2) are affine sets. [Hint: extend $\mathbf{v}_1$ to an orthonormal basis of $\mathbf{R}^n$.]