Dynamic Liquidation Under Market Impact *

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Abstract

The optimal liquidation problem with transaction costs, which includes a positive fixed cost, and market impact costs, is studied in this paper as a constrained stochastic optimal control problem. We assume that trading is instantaneous and the dynamics of the stock to be liquidated follows a geometric Brownian motion. The solution to the impulse control problem is computed at each time step by solving a linear partial differential equation and a maximization problem. In contrast to results obtained from the static formulation in Almgren and Chriss (2000), when risk is not considered, the optimal liquidation strategy from our stochastic control formulation depends on temporary market impact cost and permanent market impact cost parameters. In addition, our computational results indicate the following properties of the optimal execution strategy from the stochastic control formulation. Due to the existence of no-transaction region, it may not be optimal for some individuals to sell their assets on some trading dates. As the value of the permanent market impact parameter increases, the expected optimal amount liquidated at the terminal time increases. As the value of the quadratic temporary impact cost parameter increases, the expected optimal amount liquidated at trading times tend to be uniform, and the no-transaction region shrinks. In the presence of quadratic temporary market impact costs, in contrast to optimal strategies that result from fixed and/or proportional transaction costs alone, portfolios in the selling region are neither rebalanced into the no-transaction region nor into the sell and no-transaction interface.

Keywords: optimal liquidation, transaction cost, market impact costs, no-transaction region

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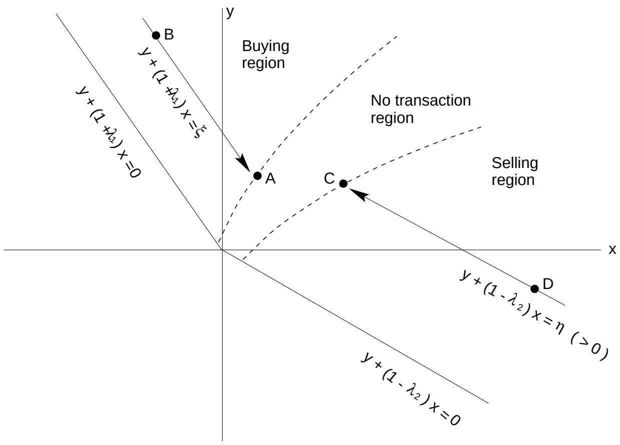
1 Introduction

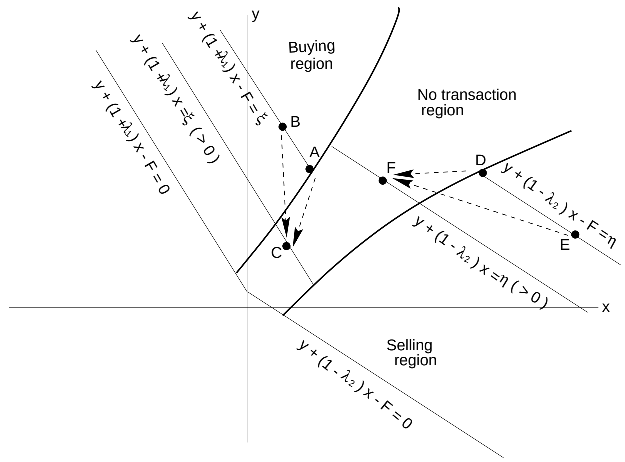
The tremendous growth in equity trading over the past decade necessitates the measurement and management of *execution costs*, see e.g., (Bertsimas and Lo, 1998). Such costs include bid/ask spreads, opportunity costs of waiting, and impact on asset prices due to trading, collectively known as market impact costs. These costs can have substantial impact on execution performance. It is well recognized that institutional large trading takes time, and that the very act of trading affects not only current proceeds from liquidation but also the price dynamics of assets which, in turn, affects future liquidation proceeds. Recent studies by Chan and Lakonishok (1995) and Keim and Madhavan (1995) show that large institutional trades are always broken up into smaller 'packages' executed over the course of several days, adapting to changing market conditions. By using a sample of 1.2 million transactions of 37 large investment firms, Chan and Lakonishok show that 20% of the market value of the packages are completed within a day and that over 53% are spread over four trading days or more.

Bertsimas and Lo (1998) study the optimal execution problem in which a trader wants to acquire stocks within a fixed time horizon. They define the best execution strategy in the sense that it provides the minimum expected cost of acquiring the stocks. In their trading model, they add an impact premium on the execution price of the trade. Such premium is modeled by a pre-defined price impact function that yields the execution price of an individual trade as function of the shares traded. When the market impact does not depend on the prevailing price of the stock, the best-execution strategy is to buy at a constant rate over the liquidation period. They have extended the analysis to the portfolio case in which trade in several securities must be executed simultaneously. Almgren and Chriss (2000) extend the work of Bertsimas and Lo (1998) by including an additional permanent impact in the model. The temporary impact refers to imbalances in supply and demand at the moment of trading caused by a trade order, the permanent impact refers to price drop that persists for the whole life of the liquidation period. Almgren and Chriss (2000) assume that trading strategy is static when they maximize the expected net proceeds for the given risk, quantified by the variance of the net expected proceeds. When one is only interested in maximizing the net proceeds, the static strategy, when the dynamics of the risky asset follows arithmetic random walk with no drift, leads to sell at a constant rate over the liquidation period. This optimal liquidation (naive) strategy is independent of the market and frictional parameters. This is not always the case, as we show in this paper, when dynamic strategies are considered in our continuous time model using stochastic optimal control theory. Also, the timings of the rebalancing are not predetermined but depend on the evolution of the stock price, which is uncertain from the initial time perspective.

In a related literature, there have been many studies that extend Merton's dynamic portfolio selection problems by incorporating various transaction costs models. Proportional transaction costs are considered, see, e.g., (Magill and Constantinides, 1976; Constantinides, 1986; Davis and Norman, 1990; Dumas and Luciano, 1991; Shreve and Soner, 1994; Akian et al., 1996; Sulem, 1997; Tourin and Zariphopoulou, 1997; Leland, 2000; Atkinson and Mokkhavesa, 2003). In addition studies of fixed transaction costs can also be found, see, e.g., (Eastham and Hastings, 1988; Hastings, 1992; Schroder, 1995; Korn, 1998). Moreover, fixed and proportional transaction



(a) Proportional Transaction Costs. B is a portfolio that lies in the buying region and A is a portfolio that lies on the buy-no transaction interface. D is a portfolio that lies in the selling region and C is a portfolio that lies on the sell-no transaction interface, λ_1 and λ_2 are the proportional buying and selling temporary market impact cost rates.



(b) Fixed and Proportional Transaction Costs. B is a portfolio that lies in the buying region, A is a portfolio that lies on the buy-no transaction interface and C is a portfolio that lies within the no transaction region. E is a portfolio that lies in the selling region, D is a portfolio that lies on the sell-no transaction interface and F is a portfolio that lies within the no transaction region.

Figure 1: Characterization of Optimal Policies

costs are considered in, e.g., (Chancelier et al., 2000; Øksendal and Sulem, 2002; Zakamouline, 2005; Chellathurai and Draviam, 2007). In the presence of proportional transaction costs, the problem is characterized by buy and no-transaction and sell and no-transaction interfaces in the portfolio space. The optimal transaction policy, in the presence of proportional transaction costs, is a minimal trading to stay inside the wedge defined by the no transaction region, preceded by an immediate transaction to the closest point in the wedge if the initial endowment is outside of it, see, Figure 1 (a). When there are strictly positive fixed and proportional transaction costs, the problem is characterized by buy and no-transaction and sell and no-transaction interfaces, and buy and sell targets in the portfolio space. If there are two portfolios which lie in the buying (selling) region so that their net values remain the same, then the risky asset is bought (sold) such that the two rebalancings result in the same buy target (sell target) portfolio with the same net value which lies within the no transaction region, see, Figure 1 (b). The buy and the sell targets coincide when the proportional transaction costs are zero. In the presence of fixed (and proportional) transaction costs, the volume of the trade is large if it takes place. More recently, Ly Vath et al. (2007) study the optimal liquidation problem with fixed transaction costs and market impact costs. They characterize the value function as the unique viscosity solution to the associated quasi-variational Hamilton-Jacobi-Bellman (HJB) inequality. Unfortunately no numerical results are provided in the paper.

In this paper, we study the dynamic liquidation problem with fixed, proportional or quadratic transaction costs, and market impact costs in a dynamic portfolio selection framework based on a continuous time stochastic optimal control approach. We compute dynamic optimal trading strategies that maximize the expected utility of the net proceeds resulting from the trading of a large block of equity over a fixed time horizon. Rebalancing is assumed to be instantaneous, and the controls are the amounts of the risky asset sold. The market impact costs incurred depend on the amount liquidated. Ly Vath et al. (2007) formulate the optimal liquidation problem with fixed transaction costs and market impact costs as an impulse control problem. In this paper, the solution to the impulse control problem is computed at each time step by solving a linear partial differential equation and a maximization problem. Computational results are provided to illustrate characteristics of optimal trading strategies. In the presence of market impact and quadratic transaction costs, the portfolio space is again divided into trading region and no-transaction region. But in the presence of quadratic transaction costs, in contrast to optimal strategies that result from fixed and/or proportional transaction costs alone, portfolios in the selling region can neither be rebalanced into the no-transaction region nor into the sell and no-transaction interface.

The presentation of the paper is organized as follows. In Section 2, a numerical method is developed for the dynamic liquidation problem when the portfolio consists of a risk-free asset, and a risky asset whose price dynamics is modeled by a geometric Brownian motion. At every point of the transaction regions, necessary conditions satisfied by the value function are derived. In Section 3, a computational algorithm is presented. At each time, the problem is reduced to solving a constrained maximization problem and a degenerate linear partial differential equation. A monotone upwind finite difference scheme is developed to discretize the partial differential equation (PDE) so that the discrete system leads to an M-matrix that guarantees the discrete maximum principle. In Section 4, we present computational results for the optimal trading strategies when the utility function for the terminal wealth is a power-law function. Section 5 concludes the paper.

2 A Computational Approach by Solving a Linear Partial Differential Equation and Maximization

Consider the problem of liquidating a large number of shares in a stock before time T to maximize some expected utility of the terminal wealth. We assume that the dynamics of the stock is characterized by

$$dS(t) = \alpha\, S(t)\, dt + \sigma\, S(t)\, dB_t \tag{1}$$

where $S(t)$ denotes the price of one share of the stock at time t, and dB_t is the increment of a Brownian motion. In (1), α is the instantaneous conditional expected return and σ^2 is the instantaneous conditional variance of the return. We also assume that there exists a risk-free asset whose dynamics is characterized by

$$dS_0(t) = r\, S_0(t)\, dt$$

where $S_0(t)$ denotes the price of one unit of the risk-free asset at time t, and r is the instantaneous rate of return from the risk-free asset, which is assumed to be constant. Let $x(t) = N(t)S(t)$ denote the value of holdings in the stock, where $N(t)$ is the number of shares of the asset held. Its dynamics, when there is no trading, is given by

$$dx(t) = \alpha\, x(t)\, dt + \sigma\, x(t)\, dB_t. \tag{2}$$

Similarly if $y(t)$ denotes the value of the holdings in the risk-free asset, its dynamics is given by

$$dy(t) = r\, y(t)\, dt. \tag{3}$$

Thus, for a small value of $\triangle t > 0$, equations (2) and (3) lead to

$$\begin{aligned} \triangle x(t) &\stackrel{\text{def}}{=} x(t+\triangle t) - x(t) = \alpha\, x(t)\, \triangle t + \sigma\, x(t)\, \triangle B_t + o(\triangle t) \\ \triangle y(t) &\stackrel{\text{def}}{=} y(t+\triangle t) - y(t) = r\, y(t)\, \triangle t + o(\triangle t), \end{aligned} \tag{4}$$

where $o(\triangle t)$ refers to higher-order terms.

Let $0 \le q(x,y,t) \le x$ denote the amount of risky asset sold at time t in state (x,y) so that the maximum amount of liquidation is x. The investor pays $\mathcal{C}(q)$ as transaction costs on the sale of the risky asset, which includes a positive fixed transaction cost ($\$F > 0$) on each selling of the risky asset. The transaction cost function is given by

$$\mathcal{C}(q) = \Lambda(q) + \lambda_s q + \lambda_q q^2 \tag{5}$$

where the impulse function $\Lambda(q)$ is defined by

$$\Lambda(q) = \begin{cases} F, & \text{if} \quad q > 0 \\ 0, & \text{if} \quad q = 0 \ . \end{cases}$$

In (5), λ_s and λ_q are non-negative parameters that characterize the temporary market impact. One would not trade if $q - \mathcal{C}(q)$ is negative. We assume that the liquidation region and the parameters λ_s and λ_q are such that $q-\mathcal{C}(q)$ remains nonnegative, and in all our numerical experiments, this condition is satisfied automatically. Also we assume that trading is instantaneous, and the value of the holdings in the equity is reduced by a factor of $(1 - \mu q)$ whenever selling takes place. The parameter μ specifies the permanent effect of selling on the price dynamics of the equity. Similar market impact functions have been considered by Ly Vath *et al.* (2007).

Let $W(T)$ denote the net terminal wealth as the amount of money in the risk-free asset resulting from the liquidation of the risky asset in $[0, T]$. The investor's objective is to maximize the expected value of the discounted utility $\phi(W(T))$ of the terminal wealth. Assume that $\rho > 0$ is a constant discount factor. The utility function of the terminal wealth is assumed to be monotonically increasing and concave in its argument to represent a risk-averse investor.

Variational inequality, which consists of equations and inequalities that characterize the selling and no-transaction regions separately, has been considered as the basic building block to characterize and analyze dynamic portfolio selection problems with fixed and proportional transaction costs, see, e.g., (Øksendal and Sulem, 2002; Zakamouline, 2005). Chen and Forsyth (2007) also solve a guaranteed minimum withdraw benefit problem based on the variational inequality formulation. Based on Bellman's principle (Bellman and Dreyfus, 1962), Ly Vath et al. (2007) recently formulate the optimal execution problem under the market impact and fixed cost as the unique viscosity solution to a quasi variational inequality problem. Assume that $J(x, y, t)$ denotes the value function which is the maximum conditional expected value of the discounted utility of the terminal wealth at T. The quasi variational inequality problem has the form of

$$\max\{MJ(x, y, t) - J(x, y, t), \ \ \mathcal{L}J(x, y, t) - \rho J(x, y, t)\} = 0, \tag{6}$$

where $\mathcal{L}$ is the associated differential operator defined by

$$\mathcal{L}J(x, y, t) \stackrel{\text{def}}{=} \frac{\partial J}{\partial t}(x, y, t) + \alpha x \frac{\partial J}{\partial x}(x, y, t) + ry\frac{\partial J}{\partial y}(x, y, t) + \frac{\sigma^2}{2}x^2\frac{\partial^2 J}{\partial x^2}(x, y, t)$$

and the intervention operator defined by

$$MJ(x, y, t) = \sup_{0<q\leq x} \{ \ J(\xi(x, q), \eta(y, q), t)\}$$

where $\xi(x, q) = (x - \ q(x, y, t))(1 - \mu q(x, y, t))$, $\eta(y, q) = y + q(x, y, t) - \mathcal{C}(q(x, y, t)$.

The above formulation and its mathematical proof are quite complex and the quasi variational inequality problem can only be solved numerically. For further details, one may refer to Ly Vath et al. (2007). In Chen and Forsyth (2007), this type of quasi variational inequality problem is solved by finite difference discretization scheme which involves solving a sequence of maximization based on

$$J(x,y,t) = \sup_{0<q\leq x} \{\, J((x-q)(1-\mu q), y+q-\mathcal{C}(q), t)\}$$

and an implicit, stable, monotone, consistent, finite difference method derived from

$$\mathcal{L}J = \rho J$$

along with the boundary conditions.

Using the intuitive notion $t+$ to denote time t immediately after trading, we provide below, based on Bellman's principle, an informal derivation for a different characterization which leads to the same computational scheme. Note that the explanation is based on the observation that, under the assumption of a positive fixed cost $F > 0$, at any x, y, t, there exists $\Delta t > 0$ such that it is optimal not to trade in $(t, t+\Delta t)$.

Let $t+$ denote immediately after trading time t and $x(t+)$ and $y(t+)$ denote the investor's holdings in the risky and risk-free assets, immediately after selling takes place at time t in state (x, y), i.e.,

$$\begin{aligned} x(t+) &= (x \; - \; q(x,y,t))\,(1-\mu q(x,y,t)) \\ y(t+) &= y \; + \; q(x,y,t) - \mathcal{C}\,(q(x,y,t)). \end{aligned}$$

Since the fixed cost $F > 0$, there exists $\Delta t > 0$ such that it is optimal not to trade in $(t, t+\Delta t)$. Using equations in (4), holdings in the assets at time $t + \triangle t$ can be written as

$$\begin{aligned} x(t+\triangle t) &= x(t+) + \alpha\; x(t+)\;\triangle t + \; \sigma\; x(t+)\;\triangle B_t + o(\triangle t) \\ &= x(t+) + \triangle x(t), && (7) \\ y(t+\triangle t) &= y(t+) + r\; y(t+)\;\triangle t + o(\triangle t) \\ &= y(t+) + \triangle y(t). && (8) \end{aligned}$$

Let $J(x, y, t)$ denote the value function which is the maximum conditional expected value of the discounted utility of the terminal wealth at T, i.e.,

$$J(x,y,t) = \sup_{\Omega_1}\; E_t\left\{ e^{-\rho(T-t)}\;\phi[\,W(T)\,]\,\right\}. \tag{9}$$

Here E_t denotes the conditional expectation operator at time t in state $x(t) = x,\; y(t) = y$. In (9), Ω_1 is the set of all integrable non-anticipative processes $0 \leq q(x,y,s) \leq x,\; t \leq s < T$, $(x, y) \in D$, where D is the liquidation region defined by

$$D = \{\; (x,y) \in R^2 \mid x \geq 0,\; y \geq 0, W > 0\;\}. \tag{10}$$

Assume that $x > 0$ and $0 \leq t < T$. Since there is no trading in $(t, t+\Delta t)$, Bellman's principle of optimality implies

$$J(x,y,t)\, e^{\rho\Delta t} \;=\; \sup_{\Omega}\; E_t\; \{\, J[\,x(t+\triangle t),\; y(t+\triangle t),\; t+\triangle t\,]\,\}.$$

where

$$\Omega = \{q(x,y,t) | 0 \le q(x,y,t) \le x, (x,y) \in D, 0 \le t < T\}.$$

Using equations (7) and (8), the above equation can be stated as

$$J(x,y,t)\, e^{\rho\Delta t} \; = \; \sup_{\Omega} \; E_t \; \{ \, J[\, x(t+) + \triangle x, \; y(t+) + \triangle y, \; t + \triangle t \,] \, \}. \tag{11}$$

Since$\Delta x \to 0$ as $\triangle t \downarrow 0$, taking the limit $\triangle t \downarrow 0$ in (11), we get

$$J(x,y,t) = \sup_{\Omega} \; \{ \; J(x(t+), y(t+), t+) \} \tag{12}$$

where $J(x(t+), y(t+), t+)$ is the utility associated with selling an amount $q(x,y,t)$ at time t in state $x(t) = x$, and $y(t) = y$.

Using Taylor's expansion and properties of the Brownian motion,

$$\begin{aligned} & E_t \; \{ \, J[\, x(t+) + \triangle x, \; y(t+) + \triangle y, \; t + \triangle t \,] \\ = \; & J(x(t+), y(t+), t+) + \triangle t \frac{\partial J}{\partial t}(x(t+), y(t+), t+) \\ & + \triangle t \left[\alpha \; x(t+) \; \frac{\partial J}{\partial x}(x(t+), y(t+), t+) + \; r \; y(t+) \; \frac{\partial J}{\partial y}(x(t+), y(t+), t+) \right. \\ & \left. + \; \frac{\sigma^2}{2} \; x(t+)^2 \; \frac{\partial^2 J}{\partial x^2}(x(t+), y(t+), t+) \right] \; + o((\triangle t)^2). \end{aligned} \tag{13}$$

From Bellman's principle and the fact that it is optimal not to trade in $(t, t + \Delta t)$,

$$J(x(t+), y(t+), t+)\, e^{\rho\Delta t} \; = \; E_t \; \{ \, J[\, x(t+) + \triangle x, \; y(t+) + \triangle y, \; t + \triangle t \,] \, \}. \tag{14}$$

From (13) and (14), letting $\triangle t \downarrow 0$, we have

$$\begin{aligned} \rho \; J(x(t+), y(t+), t+) = & \tfrac{\partial J}{\partial t}(x(t+), y(t+), t+) \; + \; \alpha \; x(t+) \; \tfrac{\partial J}{\partial x}(x(t+), y(t+), t+) \\ & + \; r \; y(t+) \; \tfrac{\partial J}{\partial y}(x(t+), y(t+), t+) \; + \; \tfrac{\sigma^2}{2} \; x(t+)^2 \; \tfrac{\partial^2 J}{\partial x^2}(x(t+), y(t+), t+). \end{aligned}$$

Let q_{opt} solves (12) and

$$\begin{aligned} x(t+)_{opt} &= (x \; - \; q_{opt}(x,y,t))\,(1 - \mu q_{opt}(x,y,t)) \\ y(t+)_{opt} &= y \; + \; q_{opt}(x,y,t) - \mathcal{C}\,(q_{opt}(x,y,t))\,. \end{aligned} \tag{15}$$

Thus the value function $J(x,y,t)$ and $q_{opt}(x,y,t)$, which solve the stochastic optimal control problem, satisfy

$$\begin{cases} \rho \; J(x,y,t+) = \tfrac{\partial J}{\partial t}(x,y,t+) \; + \; \alpha \; x \; \tfrac{\partial J}{\partial x}(x,y,t+) \\ \qquad\qquad + \; r \; y \; \tfrac{\partial J}{\partial y}(x,y,t+) \; + \; \tfrac{\sigma^2}{2} \; x^2 \; \tfrac{\partial^2 J}{\partial x^2}(x,y,t+) \\ J(x,y,t) = \sup_{\Omega} \; \{ \; J(x(t+), y(t+), t+) \}. \end{cases} \tag{16}$$

The terminal condition is given by

$$J(x, y, t = T) = \phi(w) \tag{17}$$

where $w = y + max(x - \mathcal{C}(x), 0)$ is the cash after liquidation of the risky asset.

At $x(t) = 0, 0 < t < T$, there is nothing to sell. Thus

$$x(s) = 0,\ q(x, y, s) = 0,\ x(s) = x(s+)\ \ t \leq s \leq T$$

in which case equation (11) takes the form

$$\rho\ J(x, y, t) = \frac{\partial J}{\partial t}(x, y, t)\ +\ ry\ \frac{\partial J}{\partial y}(x, y, t),\ x = 0. \tag{18}$$

To get the optimal control $q(x, y, t)$ and the associated value function $J(x, y, t)$ we have to solve (16) subject to the terminal condition (17) and the boundary condition (18), and there are no analytical solutions.

3 Computational Implementation

In this section we present a computational algorithm to determine the optimal execution strategy and the value function by iterating backwards in time based on finite difference approximations.

The characterization (16) suggests that the value function $J(x, y, t)$ and the liquidation strategy $q(x, y, t)$ can be computed numerically backwards in time by solving a linear partial differential equation followed by a maximization problem at each time step. More specifically, let $\tilde{J}(x, y, t+)$ denote the numerical approximation of the value function $J(x, y, t+)$. At each time step, we determine the value function and the associated optimal controls as follows:

1. In the liquidation region D, solve the linear PDE

$$\rho\ \tilde{J}(x, y, t+) = \frac{\partial \tilde{J}}{\partial t}(x, y, t+) + \alpha x \frac{\partial \tilde{J}}{\partial x}(x, y, t+) + ry \frac{\partial \tilde{J}}{\partial y}(x, y, t+) + \frac{\sigma^2 x^2}{2} \frac{\partial^2 \tilde{J}}{\partial x^2}(x, y, t+) \tag{19}$$

   along with the terminal and boundary conditions. For any x, y and $q(x, y, t)$, $\tilde{J}(x, y, t+)$ can be obtained from linear interpolation using a triangular element. The triangular elements are obtained from the rectangular elements with coordinates $(x_1, y_1), (x_2, y_2 = y_1), (x_3 = x_2, y_3), (x_4 = x_1, y_4 = y_3)$, by dividing into two triangular elements with coordinates $(x_1, y_1), (x_2, y_2), (x_4, y_4)$, and $(x_2, y_2), (x_3, y_3), (x_4, y_4)$.

2. At (x, y, t), solve the optimization problem

$$J(x, y, t) = \sup_{\Omega} \{\ J(x(t+), y(t+), t+)\} \tag{20}$$

   using $\tilde{J}$ in (19) to obtain the optimal control $q_{opt}(x, y, t)$.

3. At (x, y, t), using the calculated optimal control $q_{opt}(x, y, t)$, calculate

$$J(x, y, t) = J(x(t+)_{opt}, y(t+)_{opt}, t+).$$

We note again that the above computational scheme is same as the finite difference computational scheme in Chen and Forsyth (2007), which is based on the variational inequality formulation (6).

We develop an upwind finite difference scheme to discretize the PDE (19), or equivalently,

$$\rho\ J(x,y,t) = \frac{\partial J}{\partial t}(x,y,t)\ +\ a_1(x,y,t)\ \frac{\partial J}{\partial x}(x,y,t)$$

$$+\ a_2(x,y,t)\ \frac{\partial J}{\partial y}(x,y,t)\ +\ \frac{1}{2}\ b_1^2(x,y,t)\ \frac{\partial^2 J}{\partial x^2}(x,y,t), \tag{21}$$

where

$$a_1(x,y,t) = \alpha\ x,\ \ a_2(x,y,t) = r\ y,\ \ b_1(x,y,t) = \sigma\ x.$$

Assume that the value function at the time step $t + \triangle t$, $J(x, y, t + \triangle t)$, is known, where $\triangle t$ is the step size in time. At the first time step, $t = T - \triangle t$, $J(x, y, t + \triangle t)$ is the utility from terminal wealth.

The objective of the upwind discretization is that the resulting difference equations satisfy a modified partial differential equation whose solution $\tilde{J}(x, y, t)$ is twice differentiable in the liquidation region and differs from the original partial differential equation by truncation error. The determination of the modified partial differential equation, and in particular, the leading order truncation error provides essential information as to the behaviour of the numerical solution (Hirsch, 1988).

For computational purposes the PDE (21) needs to be augmented with the physical boundary conditions to get the unique solution in a localized computational domain. The localized liquidation region, D, is defined as

$$D = \{(x,y) \in R^2 | 0 \le x \le x_{max}, 0 \le y \le y_{max}, W > 0\}$$

where x_{max} and y_{max} are given parameters. The PDE (21) is parabolic in x and t and degenerate in y. The boundary conditions should be selected to replicate this behavior.

The PDE (21) itself is applied on $y = 0, 0 \le x < x_{max}$. When y_{max} is large, if the value function and its derivatives are bounded, the PDE (21) leads to, on $y = y_{max}, 0 \le x < x_{max}$,

$$\frac{\partial \tilde{J}}{\partial y} \longrightarrow 0. \tag{22}$$

Since $y = y_{max}, 0 \le x < x_{max}$, is an artificial boundary, in consistent with the PDE (21), boundary condition (22), where $y = y_{max}, 0 \le x < x_{max}, 0 \le t < T$, may be replaced by

$$\rho\ \tilde{J}(x,y,t) = \frac{\partial \tilde{J}}{\partial t}(x,y,t)\ +\ a_1(x,y,t)\ \frac{\partial \tilde{J}}{\partial x}(x,y,t)\ \ +\frac{1}{2}\ b_1^2(x,y,t)\ \frac{\partial^2 \tilde{J}}{\partial x^2}(x,y,t). \tag{23}$$

When x_{max} is large, if the value function and its derivatives are bounded, the PDE (21) leads to

$$\frac{\partial \tilde{J}}{\partial x} \longrightarrow 0, \qquad \frac{\partial^2 \tilde{J}}{\partial x^2} \longrightarrow 0. \tag{24}$$

In consistent with the PDE (21), boundary condition (24), where $x = x_{max}, 0 \leq y < y_{max}, 0 \leq t < T$, leads to

$$\rho\ \tilde{J}(x,y,t) = \frac{\partial \tilde{J}}{\partial t}(x,y,t)\ +\ a_2(x,y,t)\ \frac{\partial \tilde{J}}{\partial y}(x,y,t). \tag{25}$$

The PDE (21) is discretized on a rectangular grid of the liquidation region D. The left and right neighborhood nodes of any interior node (i,j) are indicated by the indices $(i-1,j)$ and $(i+1,j)$, respectively. The lower and upper neighborhood nodes are identified by the indices $(i,j-1)$ and $(i,j+1)$, respectively. Let (x_i, y_j) denote the co-ordinates of the interior node (i,j), and let

$$x_{i+1} - x_i = h_R, \quad x_i - x_{i-1} = h_L, \quad y_{j+1} - y_j = k_U \ \text{ and } \ y_j - y_{j-1} = k_D.$$

The subscript denoting time is not shown and it is understood that $J(x_i, y_j, t) = J_{i,j}$. Thus the numerical scheme below is fully implicit. For brevity, we use $a_1(x_i, y_j, t) = a_1$, $a_2(x_i, y_j, t) = a_2$, and $b_1(x_i, y_j, t) = b_1$.

Using standard techniques in finite difference method equation (21), along with the boundary conditions, is discretized by

$$\begin{aligned} &C_{i-1,j}\ \tilde{J}_{i-1,j}\ +\ C_{i,j-1}\ \tilde{J}_{i,j-1}\ +\ C_{i,j+1}\ \tilde{J}_{i,j+1}\ +\ C_{i+1,j}\ \tilde{J}_{i+1,j} \\ &\qquad - C_{i,j}\ \tilde{J}_{i,j}\ +\ \left(\rho + \frac{1}{\triangle t}\right)\ \tilde{J}_{i,j}\ =\ \frac{\tilde{J}(x_i, y_j, t+\triangle t)}{\triangle t}, \end{aligned} \tag{26}$$

where

$$C_{i-1,j} \ = \ \frac{a_1 - |a_1|}{2h_L} \ - \ b_1^2 \left\{ \frac{1}{(h_L + h_R)} \ \left(\frac{1}{h_L}\right) \right\}, \tag{27}$$

$$C_{i,j-1} \ = \ \frac{a_2 - |a_2|}{2k_D} \tag{28}$$

$$C_{i,j+1} \ = \ \frac{-a_2 - |a_2|}{2k_U} \tag{29}$$

$$C_{i+1,j} \ = \ \frac{-a_1 - |a_1|}{2h_R} \ - \ b_1^2 \left\{ \frac{1}{(h_L + h_R)} \ \left(\frac{1}{h_R}\right) \right\} \tag{30}$$

$$C_{i,j} \ = \ C_{i-1,j} \ + \ C_{i,j-1} \ + \ C_{i,j+1} \ + \ C_{i+1,j}\,. \tag{31}$$

The boundary conditions (18), (23), and (25) are particular cases of the PDE (15), and hence, can be discretized exactly as in the case of an interior node.

Equations (26)-(31) constitute the required discrete equations at every node in the liquidation region. In each row associated with a node, all off-diagonal entries $C_{i-1,j}$, $C_{i+1,j}$, $C_{i,j-1}$, and $C_{i,j+1}$ are non-positive and the diagonal entry is positive. Also, the sum of all the entries of each row corresponding to every interior node is positive. All these conditions satisfy the requirements for an M-matrix, which ensures the discrete maximum principle. The numerical solution cannot have oscillation at any time (Strang, 1986; Ciarlet, 1970).

It can be shown that as h_L, h_R, k_D, k_U, and $\triangle t \longrightarrow 0$, the finite difference equation (26) converges to the equation (21) defined on any interior node (i, j) at time t, and hence, the scheme is consistent. If the order of h_L^2, $h_L\, h_R$, h_R^2, k_U^2, $k_U\, k_D$, k_D^2, $\triangle t^2$ and other higher order terms are neglected, equation (26) reduces to

$$\begin{aligned}\rho\ \tilde{J}(x_i, y_j, t) \;&=\; \frac{\partial \tilde{J}}{\partial t}(x_i, y_j, t) \;+\; a_1(x_i, y_j, t)\ \frac{\partial \tilde{J}}{\partial x}(x_i, y_j, t) \\ &\quad + a_2(x_i, y_j, t)\ \frac{\partial \tilde{J}}{\partial y}(x_i, y_j, t) \;+\; \frac{b_1^{*2}}{2}(x_i, y_j, t)\ \frac{\partial^2 \tilde{J}}{\partial x^2}(x_i, y_j, t) \\ &\quad + \frac{b_2^{*2}}{2}(x_i, y_j, t)\ \frac{\partial^2 \tilde{J}}{\partial y^2}(x_i, y_j, t) \end{aligned} \tag{32}$$

where

$$\begin{aligned} b_1^{*2}(x_i, y_j, t) \;&=\; b_1^2(x_i, y_j, t) \;+\; \frac{h_R}{2}\ (\ a_1 + |a_1|\ ) \;-\; \frac{h_L}{2}\ (\ a_1 - |a_1|\ ) \\ b_2^{*2}(x_i, y_j, t) \;&=\; \frac{k_U}{2}\ (\ a_2 + |a_2|\ ) \;-\; \frac{k_D}{2}\ (\ a_2 - |a_2|\ ). \end{aligned}$$

From equation (32), we infer that if the linear terms involving h_L, h_R, k_D and k_U are included, we have some additional terms in the coefficients of $\frac{\partial^2 \tilde{J}}{\partial x^2}(x_i, y_j, t)$ and $\frac{\partial^2 \tilde{J}}{\partial y^2}(x_i, y_j, t)$. Equation (32) says that the numerical scheme (26) introduces numerical viscosity (Hirsch, 1988), $\frac{h_R}{4}\ (\ a_1 + |a_1|\ ) - \frac{h_L}{4}\ (\ a_1 - |a_1|\ )$, along with the existing diffusion term, $\frac{b_1^2(x_i, y_j, t)}{2}$, in the direction of the x-axis; and $\frac{k_U}{4}\ (\ a_2 + |a_2|\ ) - \frac{k_D}{4}\ (\ a_2 - |a_2|\ )$, in the direction of the y-axis. These additional numerical diffusion terms tend to zero as the step sizes tend to zero. The numerical scheme adds artificial viscosity ensuring that the value function (numerical) behaves smoothly at any point in the liquidation region where the value function (exact) may not be smooth in the classical sense. That is, a possible discontinuity in derivatives of J are automatically smoothened by the numerical scheme. As the step sizes go to zero, the numerical solution $\tilde{J}$ tends to the solution to the original stochastic optimal control problem. The upwind discretization procedure is widely used to solve convection dominated problems in computational fluid dynamics (Hirsch, 1988) to smooth the flow variables whose gradients change abruptly. The value function may not be differentiable in y at all points of the liquidation region. This difficulty is overcome by the 'vanishing viscosity' method, whereby the PDE is modified by adding a small diffusion term through the upwind finite-difference method. Barles and Souganidis (1991), Barles (1997) and Bardi and Capuzzo-Dolcetta (1997) have shown that it is possible to compute the solution of a Hamilton-Jacobi-Bellman equation in a robust manner by using a scheme as long as it is consistent, monotone and stable.

4 Computational Results

To illustrate, we present numerical results using the following data

$$\alpha = 0.09\ (1/year),\ \ \rho = r = 0.05\ (1/year),\ \ \sigma = 0.40\ (1/\sqrt{\text{year}}),$$

$$T = 0.02\ (year),\ \ \lambda_s = 0,\ \ F = 50\ (\$).$$

The optimal liquidation problem is solved in the computational domain

$$D = \{\ (x, y) \in R^2 \mid 0 \le x \le x_{max} = 10^8,\ 0 \le y \le y_{max} = 10^8\ \}$$

subject to the terminal and boundary conditions. The computational domain is discretized into rectangular elements, and the number of discrete nodes on both axes is 101. The maximization problem (20) is solved by a simple grid search. Numerical experiments were performed with different sets of values for x_{max} and y_{max}. Larger values for x_{max} and y_{max} have been used, and these values have negligible effects on the optimal controls and the associated value function.

The liquidation period T is approximately five days. The step-size along time axis is $\triangle t = 0.002$ (year) so that $N\triangle t = T$. This corresponds to selling the asset twice in a trading day. The utility function for the terminal wealth is

$$\phi(w) = \frac{w^\gamma}{\gamma},\ \ 0 \ne \gamma < 1,$$

where $w = y + max(x - \mathcal{C}(x), 0)$. There may be a small region in the neighborhood of $(x, y) = (0, 0)$ where there will be no trading. Considering the size of (x_{max}, y_{max}), this region falls practically within a rectangular element, and it does not affect the discreization in any way. $\gamma = 1$ corresponds to maximizing the net liquidated value of wealth at terminal time T. For a given γ, the optimal liquidation strategy, $q_{opt}(x, y, t)$, is obtained at discrete trading times $t_j = j\triangle t, 0 \le j \le (N-1)$, where $t_0 = 0$, and $t_N = T$ are the initial and terminal times respectively.

In order to test the accuracy of the computational method, the algorithm is first implemented to numerically solve the dynamic portfolio selection problem, commonly known as the Merton problem. This problem becomes a particular case of the dynamic liquidation problem when the frictional parameters are set to zero, and the liquidation constraint $x(T) = 0$ and the constraints $q \ge 0$ on the optimal amounts liquidated are eliminated.

Table 1 shows the optimal amounts sold at $t = 0$ for different time-zero portfolio compositions $(x_{init}, y_{init} = 0)$. For the given market data with the exponent of the power-law utility function $\gamma = 0.5$, the Merton line on which no rebalancing takes place is

$$y = x,$$

x_{init} (10^7)	$\triangle x = 4.0 \times 10^6$ q_{opt} (10^7)	$\triangle x = 2.0 \times 10^6$ q_{opt} (10^7)	$\triangle x = 1.0 \times 10^6$ q_{opt} (10^7)	exact q_{opt} (10^7)
4.0	2.0	2.0	2.0	2.0
4.4	2.4	2.2	2.2	2.2
4.8	2.4	2.4	2.4	2.4
5.2	2.8	2.6	2.6	2.6
5.6	2.8	2.8	2.8	2.8
6.0	3.2	3.0	3.0	3.0
6.4	3.2	3.2	3.2	3.2
6.8	3.6	3.4	3.4	3.4
7.2	3.6	3.6	3.6	3.6

$$(\lambda_s = \lambda_q = F = \mu = 0)$$

Table 1: Optimal amounts sold for different x_{init} ($y_{init} = 0$) at $t = 0$ for different step lengths $\triangle x = \triangle y$ for the Merton problem.

and the Merton number, which quantifies the proportional investment in the risky asset, is 0.5. If the time-zero composition of the portfolio is not on the Merton line $y = x$, it is optimal to rebalance so that the amounts invested in the risky and risk-free assets are the same. Table 1 shows exact optimal amounts sold at the initial time $t = 0$ and the corresponding computed optimal amounts for different values of step size $\triangle x = \triangle y$. Table 1 shows that the error in the optimal amount sold is of the order of $\frac{\triangle x}{2}$, and it is independent of the portfolio composition. Since linear interpolation function is used on a triangular element the maximum value of the value function occurs on the nodal points. As the optimization problem is solved by grid search, q_{opt} is a multiple of $\triangle x$, the step size along x-direction. This is the reason there are no additional non-zero digits in q_{opt} (Table 1).

Next we illustrate the characteristics of the dynamic optimal liquidating strategy based on Monte Carlo simulations. We graph the curve of the expected terminal wealth against the standard deviation of the terminal wealth. Assume that the value function J and the amount of optimal trading amount q have been computed for a given distribution. For each sample price path, the terminal liquidation value, $W(T)$, of any initial portfolio (x_{init}, y_{init}) at time $t = 0$ can be calculated as follows:

1. Starting from the initial time $t_0 = 0$, let $x(t_0) = x_{init}$ and $y(t_0) = y_{init}$. Calculate the optimal control $q_0\,(x(t_0), y(t_0), t_0)$ by linear interpolation.

2. For $j = 0 : N-1$

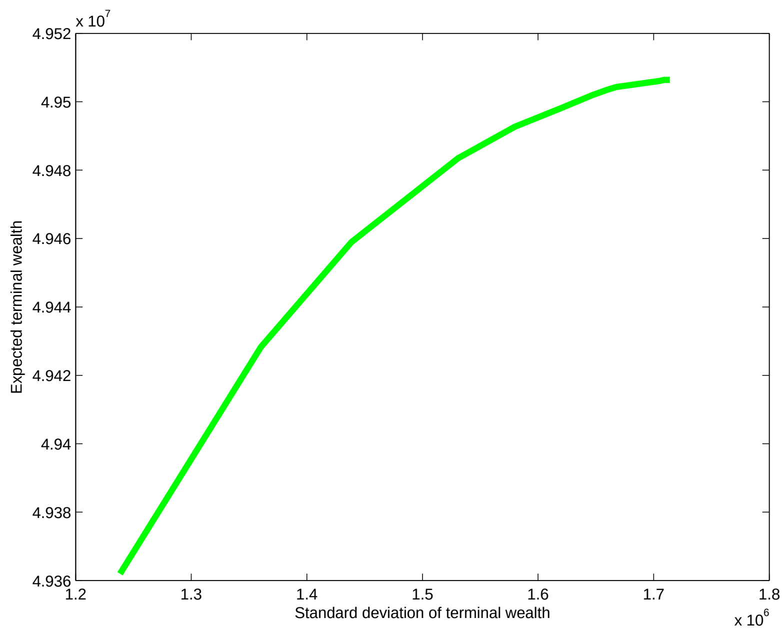
$$x(t_j+) = (x(t_j) - q_j)\,(1 - \mu q_j)\,,\ \ y(t_j+) = y(t_j) + q_j - \mathcal{C}(q_j)$$

$$x(t_j + \triangle t) = x(t_j+)\ \exp\{(\alpha - \tfrac{\sigma^2}{2})\triangle t + \sigma B_{\triangle t}\}$$

$$y(t_j + \triangle t) = y(t_j+)\ \exp\{r \triangle t\}.$$

determine q_{j+1} by linear interpolation.

3. Calculate $W(T) = y(t_N) + x(t_N) - \mathcal{C}(q_N)$.



$(\lambda_q = 2.0 \times 10^{-9},\ \mu = 10^{-10})$

Figure 2: The expected terminal wealth vs standard deviation of wealth curve: for the portfolio with initial holdings $x_{init} = 5.0 \times 10^7$, $y_{init} = 0$

Using 10,000 sample paths, we compute the expected optimal amount liquidated at any trading time $t = t_j$, and the expected value of the terminal portfolio value and its standard deviation.

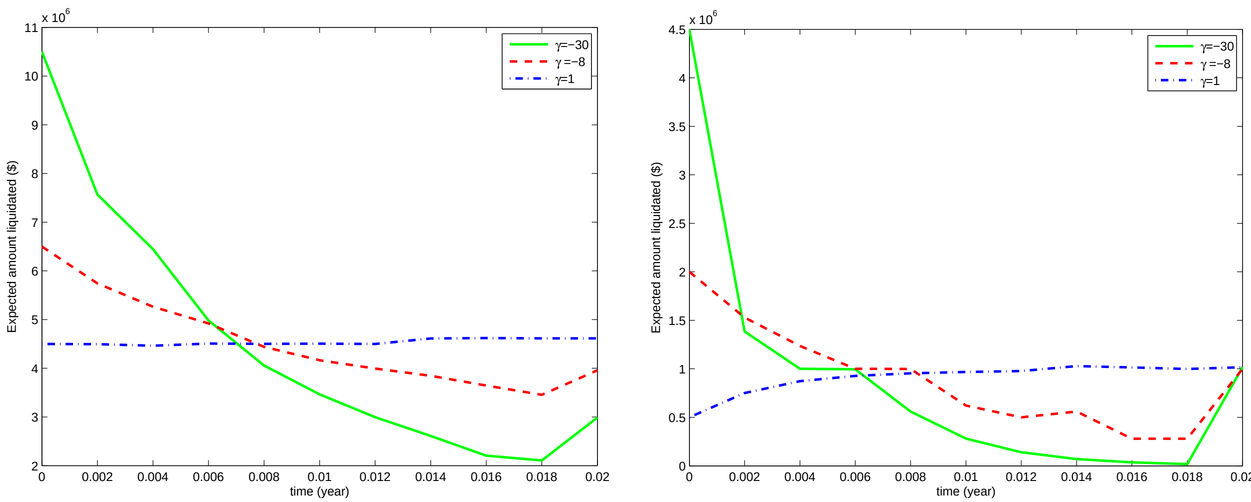
Figure 2 plots the simulated expected value of the portfolio value versus the standard deviation of the portfolio value. The initial portfolio has the time-zero initial holdings in the risky and risk-free assets given by

$$x_{init} = 5.0 \times 10^7 \ (\$), \quad y_{init} = 0 \ (\$).$$

The values of the quadratic transaction cost and market impact parameters are

$$\lambda_q = 2.0 \times 10^{-9} \ (1/\$), \quad \mu = 10^{-10} \ (1/\$).$$

The data, expected portfolio value and its standard deviation, in Figure 2 is generated by Monte-Carlo simulations for different values of γ ranging from -60 to 1.0. As expected, the marginal rate of change in expected value of terminal liquidation value decreases as risk, quantified by the standard deviation of the terminal liquidation value, increases.

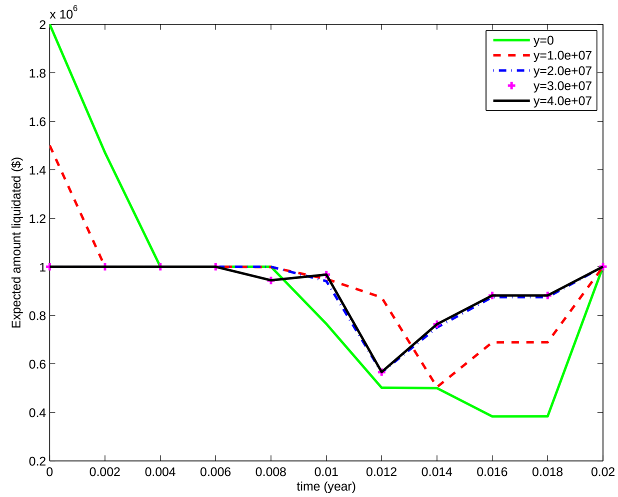


(a) $x_{init} = 5.0 \times 10^7$ (b) $x_{init} = 10^7$

$$(\ y_{init} = 0, \ \lambda_q = 2.0 \times 10^{-9}, \ \mu = 10^{-10})$$

Figure 3: Expected Optimal Amount Liquidated for Different Values of γ

Figure 3 shows the expected optimal amounts liquidated as a function of time for different values of the risk-aversion parameter γ. Figure 3 (a) corresponds to the time-zero portfolio $(x_{init}, y_{init}) = (5.0 \times 10^7, 0)$ and Figure 3 (b) is for the initial portfolio $(x_{init}, y_{init}) = (10^7, 0)$. Individuals with a high risk-aversion (e.g., the parameter $\gamma = -30$), liquidate more at time $t = 0$ than the individuals with a lower risk-aversion parameter (e.g., $\gamma = -8$). In Figure 3 (a), the initial holding in the risky asset is relatively larger than the initial risky asset holding in Figure 3 (b), the individuals who are risk-neutral (i.e., $\gamma = 1.0$) liquidate the risky asset almost uniformly over time. This is not the case for Figure 3 (b) for which the holding in the risky asset is smaller. This is in contrast to the static framework of optimal liquidation in Almgren and Chriss (2000) in which the optimal number of shares liquidated, for an individual who is risk-neutral, is uniform in time and is independent of the frictional parameters. Note that the corresponding variation in the trading amount due to asset price change should be visually negligible since the time length is very small. From Figure 3 (b), we see that, for the dynamic liquidation strategy, the optimal liquidation, for an individual with zero risk-aversion, is parameter dependent. This point will be further illustrated later.

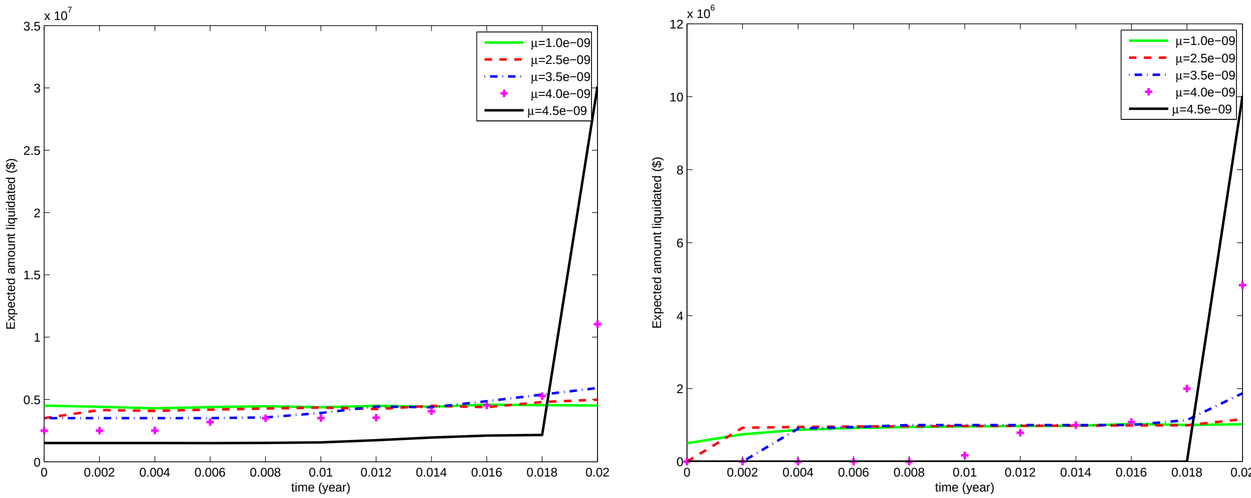


$(x_{init} = 10^7,\ \gamma = -5.0,\ \lambda_q = 2.0 \times 10^{-9},\ \mu = 10^{-10})$

Figure 4: Expected Optimal Amount Liquidated for Different Time-zero Holdings in the Risk-free Asset

Figure 4 shows different time-zero holdings in the risk-free asset lead to different expected optimal amounts liquidated at each trading time. The time-zero holdings in the asset to be liquidated is $x_{init} = 10^7(\$)$, and the holdings in the risky asset vary from 0 to $4.0 \times 10^7(\$)$. The risk aversion parameter in the power-law utility function γ is -5.0. Figure 4 shows that individuals with relatively large initial holdings in the risk-free asset (y_{init}) liquidate less at time $t = 0$ compared to an individual with zero holdings in the risk-free asset. That suggests that relatively large holding in the risk-free asset acts as a risk-tolerance parameter. This is due to the additional utility that comes from the cash, which is also observed in the Merton model (Merton, 1969).

For a fixed value of the quadratic transaction parameter $\lambda_q = 2.0 \times 10^{-9}(1/\$)$, and $\gamma = 1.0$, Figure 5 shows the expected optimal amounts liquidated at different times, for different values of the market-impact parameter, μ. Subgraphs (a) and (b) correspond to the time-zero portfolio $(x_{init}, y_{init}) = (5.0 \times 10^7, 0)$ and $(x_{init}, y_{init}) = (10^7, 0)$ respectively. As the value of the market-impact parameter increases, the expected optimal amount liquidated at the terminal time T increases. It is optimal to liquidate more at the terminal time as the liquidation at the terminal



(a) $x_{init} = 5.0 \times 10^7$

(b) $x_{init} = 10^7$

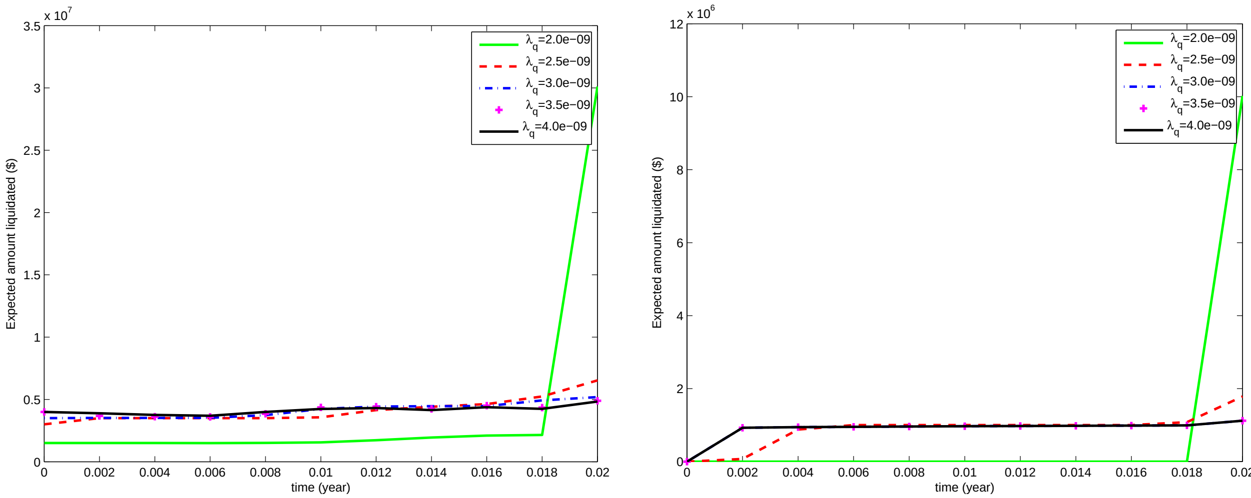
$(y_{init} = 0,\ \gamma = 1.0,\ \lambda_q = 2.0 \times 10^{-9})$

Figure 5: Expected Optimal Amount Liquidated for Different Values of Market Impact Parameter μ.

time does not affect the net portfolio value, $W(T)$. Though the risk-aversion parameter γ is unity (representing a zero risk aversion individual), the optimal strategy is not parameter independent as in the case of Almgren and Chriss (2000). Also, for the time-zero portfolio $(x_{init}, y_{init}) = (10^7, 0)$, there is no liquidation of the asset at time $t = 0$ when the market-impact parameter exceeds $\mu = 2.5 \times 10^{-9}(1/\$)$. The expected optimal amount liquidated in the first five trading times is zero for $\mu \geq 4.0 \times 10^{-9}(1/\$)$. This behavior is due to the existence of the no-transaction region, which is entirely absent in the static framework in Almgren and Chriss (2000).

For a fixed value of the market-impact parameter $\mu = 4.5 \times 10^{-9}(1/\$)$ and $\gamma = 1.0$, Figure 6 shows the expected optimal amounts liquidated as a function of time for different values of the quadratic transaction cost parameter, λ_q. Subplots (a) and (b) correspond to the time-zero portfolio $(x_{init}, y_{init}) = (5.0 \times 10^7, 0)$ and $(x_{init}, y_{init}) = (10^7, 0)$ respectively. As the value of the quadratic transaction cost parameter increases, the expected optimal amount liquidated at trading times tend to be uniform. Again, though the risk-aversion parameter γ is unity, representing a zero risk aversion individual, the optimal strategy is not parameter independent as in the case of Almgren and Chriss (2000).

Figure 7 illustrates the optimal amount liquidated as a function of holdings in the assets at $t = 0.000$, $t = 0.010$, and $t = 0.018$, respectively. The risk-aversion parameter of the power-law



(a) $x_{init} = 5.0 \times 10^7$

(b) $x_{init} = 10^7$

$(x_{init} = 5.0 \times 10^7,\ y_{init} = 0,\ \gamma = 1.0,\ \mu = 4.5 \times 10^{-9})$

Figure 6: Expected Optimal Amount Liquidated for Different Values of Quadratic Transaction cost Parameter λ_q

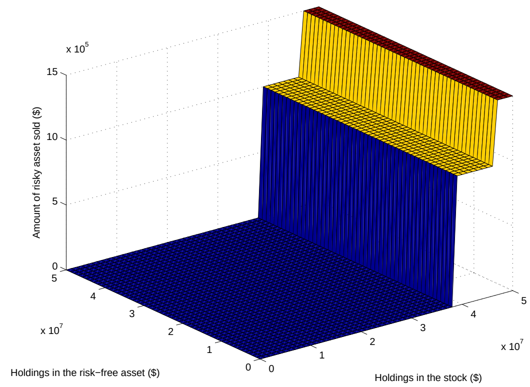
utility function, the quadratic transaction cost, and market-impact parameters are

$$\gamma = 1.0, \quad \lambda_q = 2.0 \times 10^{-9}(1/\$), \quad \mu = 4.5 \times 10^{-9}(1/\$).$$

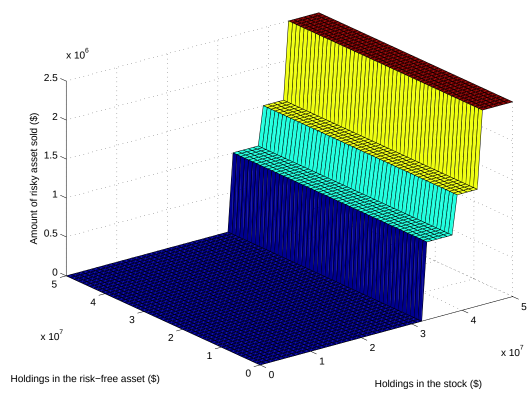
The portfolios with initial holdings in the risky asset less than $x_{init} = 2.5 \times 10^7$ liquidate only at the terminal time. Due to the existence of the no-transaction region, the expected optimal amount liquidated is not parameter independent for an individual with zero risk-aversion, quantified by $\gamma = 1.0$ (see Figures 7 and 8). In addition, Figure 7 shows that, at any trading time, the optimal liquidation strategy is dependent on the holdings of the assets. This is due to the existence of the no-transaction region, which is absent in the static framework in Almgren and Chriss (2000). Many portfolios (for example, $x = 5.0 \times 10^7$, $y \geq 0$) in the selling region are neither rebalanced into the no-transaction region nor into the sell and no-transaction interface, see Figure 7. This is the consequence of quadratic transaction costs, and this is in contrast to optimal strategies that result from fixed and/or proportional transactions alone.

To further illustrate the effect of the quadratic transaction cost and the market impact parameters, we consider larger values for λ_q and μ as follows,

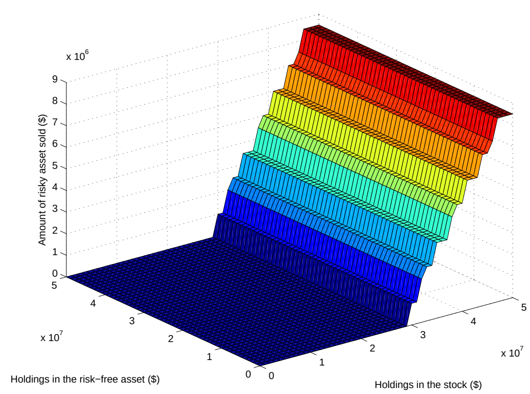
$$\lambda_q = 4.0 \times 10^{-9}(1/\$), \quad \mu = 4.5 \times 10^{-9}(1/\$).$$



(a) $t = 0$



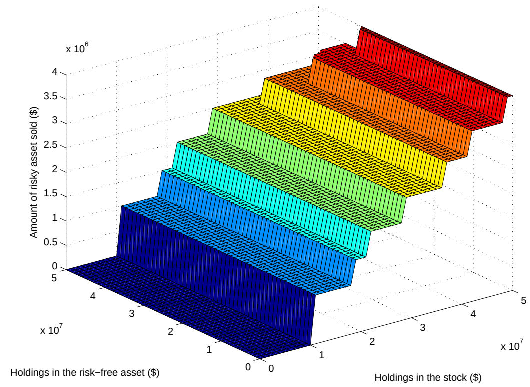
(b) $t = 0.01$



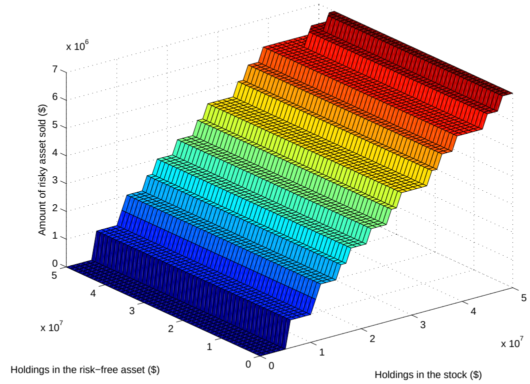
(c) $t = 0.018$

$(\gamma = 1.0,\ \lambda_q = 2.0 \times 10^{-9},\ \mu = 4.5 \times 10^{-9})$

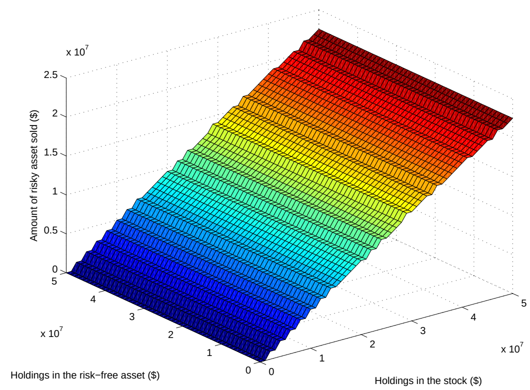
Figure 7: Optimal Amounts Liquidated



(a) $t = 0$



(b) $t = 0.01$



(c) $t = 0.018$

$(\gamma = 1.0,\ \lambda_q = 4.0 \times 10^{-9},\ \mu = 4.5 \times 10^{-9})$

Figure 8: Optimal Amounts Liquidated

As in Figure 7, the risk-aversion parameter of the power-law utility function is $\gamma = 1.0$. Figure 8 shows the optimal amounts liquidated as a function of holdings in the assets at $t = 0.000(year)$, $t = 0.010(year)$, and $t = 0.018(year)$, respectively. The no-transaction region shrinks as the quadratic transaction cost increases (for a fixed market-impact parameter). Due to the reduction of the no-transaction region, the expected optimal amount liquidated is almost uniform in time, and is parameter dependent for an individual with zero risk-aversion, quantified by $\gamma = 1.0$. Figure 8 also shows that at any trading time the optimal liquidation strategy depends on the holdings of the assets. Again, portfolios that are far away from the sell and no-transaction interface are neither rebalanced into the no-transaction region nor into the sell and no-transaction interface.

5 Conclusion

The optimal liquidation problem with transaction costs, which includes a positive fixed cost, and market impact costs, is studied in this paper as a constrained stochastic optimal control problem. We assume that trading is instantaneous and the dynamics of the stock to be liquidated follows a geometric Brownian motion. The solution to the impulse control problem is computed at each time step by solving a linear partial differential equation and a maximization problem. Our computational results indicate that the optimal liquidation strategy, corresponding to a risk-neutral individual with the power-law utility coefficient $\gamma = 1$, is dependent on frictional parameters, such as the quadratic transaction cost and market-impact parameters. This is in contrast to results obtained from the static framework of optimal liquidation in Almgren and Chriss (2000). There exists a no-transaction region in the continuous stochastic control framework; the no-transaction region is absent in the static setup in Almgren and Chriss (2000). Due to the existence of no-transaction region, it may not be optimal for some individuals to sell their assets on some trading dates. The no-transaction region affects the trading pattern more on individuals with relatively less holdings of the asset to be liquidated than on the individuals with relatively large holdings of the asset. The relatively large holdings in the risk-free asset, other things remain unchanged, acts as a risk-tolerance parameter. As the value of the market-impact parameter increases, the expected optimal amount liquidated at the terminal time increases. As the value of the quadratic transaction cost parameter increases, the expected optimal amount liquidated at trading times tend to be uniform and the no-transaction region shrinks. In the presence of the quadratic transaction cost, in contrast to optimal strategies that result from fixed and/or proportional transaction costs alone, portfolios in the selling region are neither rebalanced into the no-transaction region nor into the sell and no-transaction interface. The above differences in results are due to differences in two different model characterizations, in particular due to the nature of the resulting trading strategies (static vs dynamic).

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