Discrete hedging of American-type options using local risk minimization*†

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Abstract

Local risk minimization and total risk minimization discrete hedging have been extensively studied for European options, e.g., [11, 12]. In practice, hedging of options with American features is more relevant. For example equity linked variable annuities provide surrender benefits which are essentially embedded American options. In this paper we generalize both quadratic and piecewise linear local risk minimization hedging frameworks to American options. We illustrate that local risk minimization methods outperform delta hedging when the market is highly incomplete. In addition, compared to European options, distributions of the hedging costs are typically more skewed and heavy-tailed. Moreover, in contrast to quadratic local risk minimization, piecewise linear risk minimization hedging strategies can be significantly different, resulting in larger probabilities of small costs but also larger extreme cost.

1 Introduction

Option pricing and hedging are two of the most important problems in finance. When a market is complete, e.g., in the Black-Scholes framework [2], the option

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payoff can be perfectly replicated by continuously trading the underlying asset and a bond. Thus pricing and hedging problems are equivalent. However, markets are incomplete in practice and the risk associated with options cannot be eliminated completely. In this case option pricing becomes ambiguous since a fair value of the risk component may depend on preferences of the investors [5]. On the other hand, reducing risk as much as possible remains the main goal of hedging.

In an incomplete market, risk minimization is not completely defined until one specifies how to measure risk ([9, 12, 7, 8]). For European options, a pricing measure can be determined through quadratic risk minimization, see, e.g., [11, 12]. In this framework risk is measured by the expected quadratic difference between the payoff of an option and the value of a self-financing hedging portfolio at the moment of exercise. This is the key idea behind *total risk minimization*. However, this strategy may not always exist and may be difficult to compute, particularly under more complex asset price models.

An alternative to total risk minimization is local risk minimization. In contrast to total risk minimization, in local risk minimization the expected quadratic *incremental cost* is minimized. As can be seen for European options [3], optimal local risk minimization hedging strategies typically lead to a small total risk. Moreover, it has been shown that the choice of measure for incremental cost is important when the market is highly incomplete. For example, L_1 and L_2 norms can lead to significantly different trading strategies.

Literature in risk minimization pricing and hedging has focused on European options. However, in practice, traded options are typically of American style. For example, benefits of variable annuities in insurance are embedded with surrender options, allowing an early exit strategy. In this paper we generalize the local risk minimization hedging framework to American options. We consider the hedging problem from a writer's perspective and assume that the holder of an American option selects an optimal exercise strategy to maximize the value of the option; the manner in which the writer hedges risk is assumed to be irrelevant to the holder's exercise strategy. We show that, for American-type options, delta hedging with infrequent rebalancing can become relatively unattractive. In particular, delta hedging is more expensive and offers less protection against risk when hedging infrequently, in contrast with local risk minimization hedging. In addition, minimizing expected L_1 incremental costs, rather than minimizing expected quadratic incremental costs, can yield significantly different hedging strategies, resulting in strategies with larger probabilities of small costs but also larger extreme costs.

We first consider a simplified model in which early exercise is allowed only at hedging times. This simple model illustrates the local risk minimization framework for American-type options. Later we discuss the more complex case when hedging times and permitted early exercise times are different. Section 2 describes local risk minimization in the simplified model. Section 3 formulates local risk minimization problems in a binomial model. Section 4 describes the performance measures used. We present the computational results for the simplified model in section 5. Section 6 shows how a general model can be formulated. We conclude in section 7.

2 A simplified model

Consider a standard American-type option with strike price K and expiration time T. In this paper, an American option refers to an option which can be exercised at any time specified in the assumed asset price model. A Bermudan option refers to an option which can be exercised at a subset of times. Without loss of generality, we assume that the asset does not pay any dividends. We describe here a local risk minimization framework for American-type put options; a formulation for call options is similar.

Let $X_{(i)}$ denote the discounted asset price at time $\tau \cdot i$, where $i = 0, 1, \ldots, N$ and $\tau > 0$ is a constant satisfying $T = N \cdot \tau$. Assume that asset price $X_{(i)}$, $i = 0, 1, \ldots, N$, is an adapted process on a probability space $(\Omega, \mathcal{F}, P)$.

Consider the problem of hedging a put option with strike price K and expiry T. Assume that only a finite number of hedging times, $t_0^H = 0 < t_1^H < \cdots < t_M^H = T$, are permitted. For simplicity we suppose that these times are spaced evenly, i.e., $t_k^H = k\tau_H$ and the asset price at hedging time t_k^H is simply denoted by X_k, $k = 0, 1, \ldots, M$. In this presentation, it is assumed for simplicity that N/M is an integer and $N \geq M$. We consider the filtration $(\mathcal{F}_k)_{k=0,1,\cdots,M}$ given by $\mathcal{F}_k = \sigma(X_j | j \leq k)$, which is the σ-field generated by the variables $X_0, \cdots, X_k$.

Let t_k^E, $k = 0, \ldots, L$, denote exercise times permitted by a Bermudan option. To avoid unnecessary complexity, we consider first *a simplified model* in which hedging is permitted only at the early exercise times, i.e., $t_k^E = t_k^H$. This assumption is made through sections 2-5. The more general case will be presented in section 6.

If the option holder exercises the option at time $t_k^H \in [0, T]$, the discounted payoff H_k is given by the formula:

$$H_k = \max(0, X_k - e^{-rt_k^H} K)$$

where $r > 0$ is the constant interest rate.

We assume that the holder of an American option selects an optimal exercise strategy to maximize the value of the option; the manner in which the writer hedges risk is irrelevant to the exercise strategy. In addition, we consider the hedging problem from a writer's perspective, assuming that the exercise strategy is chosen to maximize the value to the holder. Hence it is assumed that the decision of the option holder to exercise early at time t depends only on the value of asset price X_t: the early exercise occurs as soon as $X_{(i)}$ falls below a critical value $\bar{X}_{(i)}$. The set $\{(i, \bar{X}_{(i)}), \quad i = 0, 1, \ldots, N\}$ is the set of the early exercise critical values; we discuss in §3.4 methods for determining it.

Denote a stopping moment as $M^* = \min(\{k : k \in \{0, \ldots, M\}$ and $X_k \leq \bar{X}_k\})$. This corresponds to the hedging moment at which the early exercise occurs. The corresponding stopping time will then be $\tau_H M^*$.

A *trading strategy* is given by two stochastic processes $(\xi_k)_{k=0,\ldots,M^*}$ and $(\eta_k)_{k=0,\ldots,M^*}$, adapted to the filtration $\{\mathcal{F}_k\}$, where ξ_k is the number of shares held at time t_k^H, and η_k is the amount invested in the bond at time t_k^H. Let $\xi_{M^*} = 0$, which means that we liquidate our portfolio at the stopping moment.

The value of the portfolio at time t_k, $k = 0, 1, \ldots, M^*$, is given by

$$V_k = \xi_k X_k + \eta_k$$

Denote $G_k = \sum_{j=0}^{k-1} \xi_j (X_{j+1} - X_j),\ k = 1, \ldots, M^*$. Hence G_k is the *accumulated gain* due to changes of asset prices up to time t_k^H. At time moment 0, G_0 is set to zero.

The *cumulative cost* C_k is then given by

$$C_k = V_k - G_k, \quad k = 0, \ldots, M^*.$$

A strategy is *self-financing* if its cumulative cost process $(C_k)_{k=0,1,\ldots,M^*}$ is constant over time: $C_0 = C_1 = \cdots = C_{M^*}$.

A market is complete if any claim H_{M^*} is attainable, i.e., there is a self-financing strategy with value V_{M^*} equal to H_{M^*} almost surely. If the market is incomplete, as in the case of discrete hedging, a claim is generally non-attainable and a hedging strategy has to be chosen based on some measure of optimality.

In *total risk minimization*, one tries to minimize the deviation of the self-financing portfolio value V_{M^*} from the claim H_{M^*}. Usually this is done by considering the expected value of the quadratic difference:

$$E[(H_{M^*} - V_{M^*})^2]$$

However, a solution to this minimization problem may not exist [11]. Although efficient algorithms have been developed to solve this problem (e.g., see [6]), solutions can be hard to compute for general asset models, and especially if one considers a different norm (e.g., 1-norm) to measure the difference between the portfolio value and the claim. As an alternative, a *local risk minimization* hedging strategy, which is meaningful in a hedging context and is easier to compute, can be considered.

In local risk minimization for European options, the incremental cost $\Delta C_k = C_{k+1} - C_k$ is minimized at each hedging time $k = M-1, \ldots, 0$. The final condition is $V_M = H_M, \xi_M = 0, \eta_M = H_M$. For American-type options, the incremental cost needs to be minimized at a hedging time *if* the option has not been exercised.

Since the change $(C_{k+1} - C_k)|\mathcal{F}_k$ is a random variable, there are different measures for its associated risk. In most of literature on pricing and hedging in an incomplete market, e.g., [11, 12], the expected quadratic incremental cost is minimized. This approach is called the *quadratic local risk minimization.*

We generalize the local risk minimization for European options to American-type options as follows.

At a stopping moment, $V_{M^*} = H_{M^*}$, $\xi_{M^*} = 0$ and $\eta_{M^*} = V_{M^*}$.

If $k < M^*$, the quadratic local risk minimization problem at hedging moment t_k^H is

$$\min_{\xi_k, \eta_k} E\left[(C_{k+1} - C_k)^2 | \mathcal{F}_k\right] \tag{1}$$

There are many ways to measure incremental costs; therefore it is natural to wonder how different measures will affect the resulting determination of fair values and hedging strategies. For European options, this has been investigated in [3]. In this paper, we investigate this for American options.

We similarly consider a *piecewise linear local risk minimization problem* for American-type options: at any hedging moment t_k, $(k = 0, \ldots, M-1)$, if $k < M^*$,

$$\min_{\xi_k, \eta_k} E\left[|C_{k+1} - C_k| | \mathcal{F}_k\right] \tag{2}$$

The quadratic local risk minimization produces a *mean-self-financing* strategy, i.e., $E\left[(C_{k+1} - C_k)|\mathcal{F}_k\right] = 0$. This is not the case for the piecewise linear local risk minimization. However, we can enforce the hedging strategy to be mean-self-financed and consider a *constrained piecewise linear local risk minimization problem*: at any hedging moment t_k, $k = 0, \ldots, M-1$, if $k < M^*$, we determine hedging positions ξ_k, η_k from

$$\begin{aligned} &\min_{\xi_k, \eta_k} E\left[|C_{k+1} - C_k| | \mathcal{F}_k\right] \\ \text{s.t.} \quad &E\left[(C_{k+1} - C_k)|\mathcal{F}_k\right] = 0 \end{aligned} \tag{3}$$

A local risk minimization hedging strategy can be computed via stepping backwards $k = M-1, \ldots, 0$. Next we discuss in greater detail the local risk minimization approach for American-type options in a binomial model. In particular, we discuss determination of critical values which define the holder's early exercise strategy.

3 Local risk minimization in a binomial model

To further illustrate local risk minimization hedging strategies, we consider a discrete asset price process described by a binomial tree, which can be a discrete approximation to a continuous price process satisfying a stochastic differential equation,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dZ_t,$$

where Z_t is a standard Brownian process [4].

Let us model the underlying stock price over time period $[0, T]$ using a binomial tree with N periods. Denote $\tau = T/N$. At the node (i, j) of the binomial tree, the time moment is $t_i^B = i\tau$ and the stock price is $S^j_{t_i^B} = u^{2j-i} S_0$ for $i = 0, \ldots, N,\ j = 0, \ldots, i$, where $u = 1/d = e^{\sigma\sqrt{\tau}}$. The discounted stock price is $X^j_{t_i^B} = e^{-ri\tau} u^{2j-i} X_0$. For simplicity, $X_{(i)}$ denotes $X_{t_i^B}$ and $S_{(i)}$ denotes $S_{t_i^B}$ as before.

For each node the (discounted) stock price goes up with probability p and down with probability $1-p$, where $p = \frac{e^{\mu\tau}-d}{u-d}$.

At each node (i, j) early exercise occurs if $X^j_{(i)} \leq \bar{X}_{(i)}$, where $\bar{X}_{(i)}$ denotes the critical value for early exercise.

Nodes of the binomial tree corresponding to hedging times are called hedging nodes. Hedging node $[k, j]$ is the same as node $(k \cdot n_H, j)$ in the binomial tree.

To summarize, there can be four different meanings when we refer to time:

- Continuous time, $t \in [0, T]$.
- Time in the binomial tree, $t_i^B = i \cdot \tau \quad i = 0, \ldots, N$.
- Discrete hedging times, $t_k^H = k \cdot \tau_H \quad k = 0, \ldots, M$.
- Early exercising times, $t_k^E = k \cdot \tau_E \quad k = 0, \ldots, L$.

European options correspond to the special case when $\tau_E = T$ and $n_E = N$.

In §3.4, we will discuss determination of critical exercise values. In order to describe local risk minimization calculations, for the next three subsections we assume that the critical exercise values $\bar{X}_k$ are pre-computed for each hedging moment k.

The local risk minimization hedging strategy for American-type options can be computed by backward iterations: for $k = M-1, \ldots, 0$

- Set $\xi_{k+1} = 0$, $\eta_{k+1} = H_{k+1}$ if $X_{k+1} < \bar{X}_{k+1}$.
- Compute ξ_k and η_k by minimizing incremental cost with the specified measure.

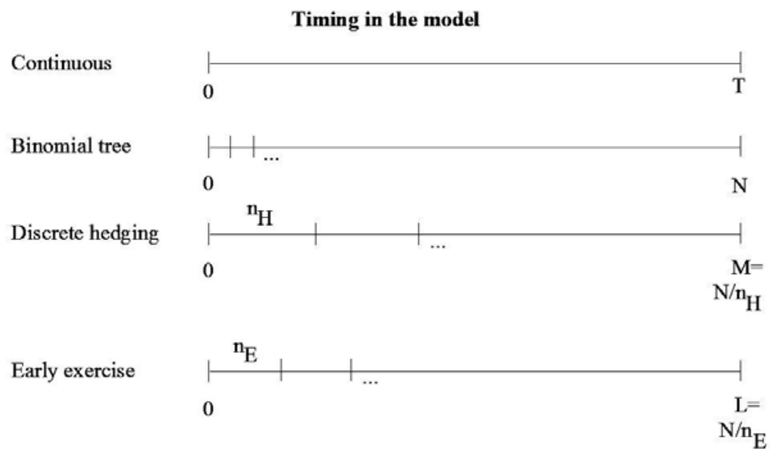


Figure 1: Timing in the model.

We now discuss computation in greater detail for each measure of incremental risk.

3.1 Quadratic local risk minimization

At time t_k^H with binomial tree state $j \in \{0, ..., k \cdot n_H\}$, the objective function in the quadratic risk minimization (1) can be written as

$$E\left[(C_{k+1} - C_k)^2 | \mathcal{F}_k\right] = E\left[(C_{k+1} - C_k)^2 | X_k = X_k^j\right] =$$

$$\sum_{\substack{l=0 \\ X_{k+1}^{j+l} > \bar{X}_{k+1}}}^{n_H} p_l \left(X_{k+1}^{j+l}(\xi_{k+1}^{j+l} - \xi_k^j) + (\eta_{k+1}^{j+l} - \eta_k^j)\right)^2 + \sum_{\substack{l=0 \\ X_{k+1}^{j+l} \leq \bar{X}_{k+1}}}^{n_H} p_l \left(H_{k+1}^{j+l} - X_{k+1}^{j+l}\xi_k^j - \eta_k^j\right)^2$$

The first sum corresponds to situations when the holder does not exercise at t_{k+1}^H, and the second when the holder does.

In a binomial model, we can write the expected quadratic incremental cost in a more compact form by introducing a 2-column matrix A, a vectors b, and a vector x defined by

$$x = \begin{pmatrix} \eta_k^j \\ \xi_k^j \end{pmatrix}, \quad A = \begin{bmatrix} \sqrt{p_0} & \sqrt{p_0} X_{k+1}^j \\ \vdots & \vdots \\ \sqrt{p_{n_H}} & \sqrt{p_{n_H}} X_{k+1}^{j+n_H} \end{bmatrix}$$

$$b = \begin{pmatrix} \sqrt{p_0}((1 - \pi_{k+1}^j)(X_{k+1}^j \xi_{k+1}^j + \eta_{k+1}^j) + \pi_{k+1}^j H_{k+1}^j) \\ \vdots \\ \sqrt{p_{n_H}}((1 - \pi_{k+1}^{j+n_H})(X_{k+1}^{j+n_H} \xi_{k+1}^{j+n_H} + \eta_{k+1}^{j+n_H}) + \pi_{k+1}^{j+n_H} H_{k+1}^{j+n_H}) \end{pmatrix}$$

where $\pi_k^j = \mathbb{I}(X_k^j \leq \bar{X}_k)$, indicating early exercise when $\mathbb{I}(X_k^j \leq \bar{X}_k) = 1$. The function $\mathbb{I}(\cdot)$ is the indicator function which equals one when the argument has a nonzero value.

The corresponding quadratic local risk minimization problem can now be written as

$$\min_{x \in \mathbb{R}^2} \|Ax - b\|_2$$

Since all prices $X_{k+1}^j, \ldots, X_{k+1}^{j+n_H}$ are different, A has full column rank. Thus, the solution can be written explicitly[1]: $x^* = (A^T A)^{-1} A^T b$.

The quadratic local risk minimization problem has an analytical solution in general and properties of the optimal hedging strategy can be examined. For example, the optimal hedging strategy is *mean-self-financing*, i.e.,

$$E\left[(C_{k+1} - C_k)|\mathcal{F}_k\right] = 0. \tag{4}$$

Equation (4) can be established by differentiating the objective function.

3.2 Piecewise linear local risk minimization

At time moment t_k, binomial tree state j, the objective function in the piecewise local risk minimization (2) can be formulated as:

$$\begin{aligned} & E\left[|C_{k+1} - C_k|\;|\mathcal{F}_k\right] = E\left[|C_{k+1} - C_k|\;|X_k = X_k^j\right] \\ = & \sum_{\substack{l=0 \\ X_{k+1}^{j+l} > \bar{X}_{k+1}}}^{n_H} p_l \left| X_{k+1}^{j+l}(\xi_{k+1}^{j+l} - \xi_k^j) + (\eta_{k+1}^{j+l} - \eta_k^j) \right| + \sum_{\substack{l=0 \\ X_{k+1}^{j+l} \leq \bar{X}_{k+1}}}^{n_H} p_l \left| H_{k+1}^{j+l} - X_{k+1}^{j+l}\xi_k^j - \eta_k^j \right| \end{aligned} \tag{5}$$

Similarly we can write (5) in a more compact form by introducing matrix a 2-column matrix A, a vector b, and a vector x defined by

$$x = \begin{pmatrix} \eta_k^j \\ \xi_k^j \end{pmatrix}, \quad A = \begin{bmatrix} p_0 & p_0 X_{k+1}^j \\ \vdots & \vdots \\ p_{n_H} & p_{n_H} X_{k+1}^{j+n_H} \end{bmatrix}$$

$$b = \begin{pmatrix} p_0((1 - \pi_{k+1}^j)(X_{k+1}^j \xi_{k+1}^j + \eta_{k+1}^j) + \pi_{k+1}^j H_{k+1}^j) \\ \vdots \\ p_{n_H}((1 - \pi_{k+1}^{j+n_H})(X_{k+1}^{j+n_H} \xi_{k+1}^{j+n_H} + \eta_{k+1}^{j+n_H}) + \pi_{k+1}^{j+n_H} H_{k+1}^{j+n_H}) \end{pmatrix}$$

where $\pi_k^j = \mathbb{I}(X_k^j \leq \bar{X}_k)$.

[1]This expression is not a good way of actually computing the solution. See [1] for more discussion.

In a binomial model, the piecewise linear local risk minimization problem can be written as

$$\min_{x \in \mathbb{R}^2} \|Ax - b\|_1$$

Unfortunately, there is no analytical solution to this problem. However, the solution can be found numerically using the approach described in [10].

In addition, the resulting trading strategy is not mean-self-financing in general. However, we can impose the mean-self-financing condition and consider the *constrained piecewise linear local risk minimization problem.*

3.3 Constrained piecewise linear local risk minimization

For the constrained piecewise linear local risk minimization problem (3), the constraint can be rewritten as:

$$E\left[C_{k+1} - C_k \,|\mathcal{F}_k\right] = E\left[C_{k+1} - C_k \,|X_k = X_k^j\right]$$

$$= \sum_{\substack{l=0 \\ X_{k+1}^{j+l} > \bar{X}_{k+1}}}^{n_H} p_l \left(X_{k+1}^{j+l}(\xi_{k+1}^{j+l} - \xi_k^j) + (\eta_{k+1}^{j+l} - \eta_k^j)\right) + \sum_{\substack{l=0 \\ X_{k+1}^{j+l} \leq \bar{X}_{k+1}}}^{n_H} p_l \left(H_{k+1}^{j+l} - X_{k+1}^{j+l}\xi_k^j - \eta_k^j\right)$$

$$= \sum_{l=0}^{n_H} p_l \left((1 - \pi_{k+1}^{j+l})(X_{k+1}^{j+l}\xi_{k+1}^{j+l} + \eta_{k+1}^{j+l}) + \pi_{k+1}^{j+l} H_{k+1}^{j+l} - X_{k+1}^{j+l}\xi_k^j\right) - \eta_k^j = 0$$

Substituting the expression for η_k^j into the objective function results in a one-dimensional L_1 problem for ξ_k^j.

3.4 Critical Values for the Early Exercising Decision

The formulation of hedging strategies for a writer has been described assuming the holder exercises a put option when the asset price falls below a critical value. We now describe how these critical values are determined.

In a complete market, no-arbitrage pricing uniquely determines the early exercise strategy: the holder exercises optimally to maximize the no-arbitrage value. In an incomplete market, no-arbitrage value is not unique even for the European option. In this paper, we are only focusing on the hedging problem for a writer. What is the exercise strategy a writer should assume when determining hedging strategies?

From the point of view of the option holder, he has a contract that permits exercising every n_E periods. Thus the holder will choose an exercise strategy to maximize the option value to him; hedging decisions of the writer are irrelevant to his exercise decision. In absence of an objective view on fair value pricing in an incomplete market, let us assume that the holder uses the initial hedging cost or the mean cumulative cost of a local risk minimization strategy as a guideline to fair values, then the maximum value to the holder will correspond to hedging every period, since initial hedging costs and average cumulative hedging costs seem to increase as hedging frequency increases. This is illustrated in Tables 3.1 and 3.2 for the European option, see [3].

Table 3.1: Initial Costs for European Puts ($n_E = N$)

IniCost			n_H						
			1	5	25	50	100	300	600
K	90	L_1	1.4254	0.0299	0.6442	0.0837	0	0	0
		L_2	1.4254	1.4204	1.3962	1.3669	1.3118	1.1348	0.9671
		L_1c	1.4254	1.3718	1.3057	1.264	1.203	0.8722	0.6516
		Δ	1.4254	1.4254	1.4254	1.4254	1.4254	1.4254	1.4254
	100	L_1	3.7499	0.5544	2.3361	0.8783	0.7925	0	0
		L_2	3.7499	3.7422	3.7035	3.6557	3.5626	3.2321	2.8703
		L_1c	3.7499	3.6674	3.5739	3.5236	3.4766	2.8802	2.2359
		Δ	3.7499	3.7499	3.7499	3.7499	3.7499	3.7499	3.7499
	110	L_1	7.7139	3.3123	6.1464	4.2234	4.1045	5.2699	2.6164
		L_2	7.7139	7.7046	7.6583	7.6	7.4833	7.0297	6.4581
		L_1c	7.7139	7.618	7.5224	7.4866	7.4951	6.9909	5.9606
		Δ	7.7139	7.7139	7.7139	7.7139	7.7139	7.7139	7.7139

When constructing a hedging strategy for the writer, we assume that the holder determines the optimal exercise strategy by evaluating the fair value under the assumption that hedging can be done at every time period (as if the market is complete in the binomial hedging model). We can construct the early exercise critical values using the binomial tree representation. Define $\mathcal{E}_{(i)} = \mathbb{I}(i = m \cdot n_E$ for some positive integer $m)$, i.e., the indicator that the early exercise can occur at the stage i. For each node (i, j) in the binomial tree define the discounted value of the option (from the point of view of policyholder) $V_{(i)}^j$ in the following way. The discounted final value at moment T should be equal to the discounted payoff: $V_{(N)}^j = V_T^j = (e^{-rT}K - X_T^j)^+$, $j = 0, \ldots, N$. Naturally, set $\bar{X}_{(N)} = e^{-rT}K$.

For node (i, j), $i = N-1, \ldots, 0$, $j = 0, \ldots, i$, calculate option value as the maximum of the continuation value $CV_{(i)}^j = p^* V_{(i+1)}^{j+1} + (1 - p^*)V_{(i+1)}^j$ and the payoff (if early exercise is allowed):

$$V_{(i)}^j = \max\left(CV_{(i)}^j,\ H_{(i)}^j \mathcal{E}_{(i)}\right),$$

where $p^* = (e^{r\tau} - d)/(u - d)$ is the risk neutral probability.

Table 3.2: Average Cumulative Costs for European Puts ($n_E = N$)

AvgCumCost			n_H						
			1	5	25	50	100	300	600
K	90	L_1	1.43	0.92	1.18	0.91	0.82	0.62	0.52
		L_2	1.43	1.42	1.39	1.37	1.31	1.13	0.97
		L_1c	1.43	1.37	1.3	1.26	1.2	0.87	0.65
		Δ	1.43	1.43	1.44	1.46	1.5	1.64	1.89
	100	L_1	3.75	3.16	3.5	3.12	2.99	2.09	1.66
		L_2	3.75	3.74	3.7	3.65	3.56	3.23	2.87
		L_1c	3.75	3.67	3.57	3.52	3.48	2.88	2.24
		Δ	3.75	3.75	3.78	3.81	3.87	4.13	4.55
	110	L_1	7.71	7.59	7.62	7.49	7.38	6.9	4.98
		L_2	7.71	7.7	7.65	7.6	7.47	7.02	6.45
		L_1c	7.71	7.62	7.51	7.49	7.48	6.98	5.96
		Δ	7.71	7.72	7.75	7.79	7.86	8.19	8.73

The state $j' = \min_j(CV^j_{(i)} > H^j_{(i)})$, at which the switch from the payoff to the continuation value occurs, determines the early exercise critical value: $\bar{X}_{(i)} = X^{j'-1}_{(i)}$.[2]

If the payoff is less than the continuation value for any j or the early exercise is not allowed at this time moment, we set $\bar{X}_{(i)} = 0$.

3.5 Delta-hedging strategy

Delta-hedging is a popular method for hedging an option, and we consider it for comparison purposes. In a binomial model, delta hedging positions at binomial tree nodes can be computed in a standard way:

$$\Delta^j_{(i)} = \frac{V^{j+1}_{(i+1)} - V^j_{(i+1)}}{X^{j+1}_{(i+1)} - X^j_{(i+1)}}, \quad i = 0, \ldots, N-1, j = 0, \ldots, i$$

This computation proceeds backwards in time. At each time, delta is computed for each node with the underlying stock price above the early exercise critical value. We can then use these delta values to construct the delta-hedging strategy: at hedging times $t^H_k, k = 0, \ldots, M-1$, the number of shares in the underlying stock is

$$\xi^j_k = \Delta^j_k.$$

[2] In the actual implementation, we take the mid-point between two binomial nodes (rather than the lowest one) to avoid numerical issues, i.e., we take $\bar{X}_{(i)} = \frac{1}{2}(X^{j'}_{(i)} + X^{j'-1}_{(i)})$.

4 Performance measures

In this section, we compare the hedging performance of different hedging strategies. In [3], performance of local risk minimization strategies was measured by *cumulative cost* and *incremental risk*.

The definitions of these random variables can be naturally extended to our model. We have already introduced the final cumulative cost C_{M^*}, which can be expressed in terms of holdings in the underlying stock as

$$C_{M^*} = H_{M^*} - \sum_{k=0}^{M^*-1} \xi_k (X_{k+1} - X_k).$$

Similarly, we consider incremental risk (average per period) as

$$IR = \frac{1}{M^*} \sum_{k=0}^{M^*-1} |C_{k+1} - C_k|$$

For each of these random variables we can find the mean, standard deviation, quantiles (including the median), skewness, kurtosis, etc. by simulation techniques. Moreover, the expected values in the binomial model can be obtained explicitly as follows.

For each hedging node of the binomial tree $[k, s]$ we can compute

- the distribution of change in cost conditioned on the stock price at hedging node $\nu = [k, s]$, i.e., $\Delta C_k | [k, s] = (C_{k+1} - C_k) | [k, s]$.

- probability $Pr(\nu)$ of reaching hedging node $\nu = [k, s]$ if $k < M^*$. We can compute $Pr(\nu)$ for each node ν using a recursion:

  $$Pr([0,0]) = 1, Pr([k{+}1, s]) = \mathbb{I}(X^s_{k+1} > \bar{X}_{k+1}) \sum_{\bar{s}=0}^{kn_H} Pr([k, \bar{s}]) Pr\{[k{+}1, s] | [k, \bar{s}]\}$$

  for $k = 0, \ldots, M-1, \ s = 0, \ldots, (k+1)n_H$.

Note that

$$\{\omega : k < M^*\} = \{\omega : \text{ hedging node } [k, s] \text{ is reached for which } X^s_k > \bar{X}_k\}$$

Hence the expected cumulative cost is:

$$E(C_{M^*}) = E\left[C_0 + \sum_{k=0}^{M^*-1} \Delta C_k\right] = C_0 + E\left[\sum_{k=0}^{M-1} \Delta C_k \mathbb{I}(k < M^*)\right] =$$

$$C_0 + \sum_{k=0}^{M-1} E\left[\Delta C_k \mathbb{I}(k < M^*)\right] = C_0 + \sum_{k=0}^{M-1} E\left[E[\Delta C_k | k < M^*]\right] =$$

$$C_0 + \sum_{k=0}^{M-1} \sum_{s: X_k^s > \bar{X}_k} E[\Delta C_k | [k,s]] P([k,s]).$$

The expected incremental risk can be calculated in the following fashion.

Let random variable ϵ represent the node of the binomial tree at which the exercise occurs. Let $\nu = [\bar{k}, \bar{s}]$ be the last hedging node visited before exercising at ϵ.

Then,

$$E\left[\frac{1}{M^*} \sum_{k=0}^{M^*-1} |\Delta C_k|\right] = E\left[E\left[\frac{1}{M^*} \sum_{k=0}^{M^*-1} |\Delta C_k| \middle| \nu\right]\right] = E\left[\frac{1}{\bar{k}+1} E\left[\sum_{k=0}^{\bar{k}} |\Delta C_k| \middle| \nu\right]\right].$$

For calculating the conditional expectation inside the expression, we use a recursive formula:

$$\rho(\nu) = E\left[\sum_{k=0}^{\bar{k}} |\Delta C_k| \middle| \nu\right] = E\left[\sum_{k=0}^{\bar{k}-1} |\Delta C_k| + |\Delta C_{\bar{k}}| \middle| \nu\right] =$$

$$E\left[E\left[\sum_{k=0}^{\bar{k}-1} |\Delta C_k| + |\Delta C_{\bar{k}}| \middle| \nu^-, \nu\right] \middle| \nu\right],$$

where ν^- is the hedging node visited before ν.

Given ν^-, the distribution of ΔC_k for $k < \bar{k} - 1$ is not affected by ν. Thus $\rho(\nu)$ is equal to

$$\rho(\nu) = E\left[E\left[\sum_{k=0}^{\bar{k}-1} |\Delta C_k| \middle| \nu^-\right] + E\left[|\Delta C_{\bar{k}}| | \nu^-, \nu\right] \middle| \nu\right] =$$

$$E\left[\rho(\nu^-) + E\left[|\Delta C_{\bar{k}}| | \nu^-, \nu\right] \middle| \nu\right]$$

Obviously $\rho([0,0]) = 0$. Note that $[|\Delta C_{\bar{k}}| | \nu^-, \nu]$ is deterministic since it represents a change in cost while going from node ν^- to ν. Since the probability of reaching each node is precomputed, we can write down each of the conditional expectations explicitly:

$$\rho(\nu) = \sum_{\substack{\text{for all } s,\\ \nu^- = [\bar{k}-1, s]}} \left(\rho(\nu^-) + \left[|\Delta C_{\bar{k}}|| \, \nu^-, \nu\right]\right) Pr(\nu^- | \nu) =$$

$$\sum_{\substack{\text{for all } s,\\ \nu^- = [\bar{k}-1, s]}} \left(\rho(\nu^-) + \left[|\Delta C_{\bar{k}}|| \, \nu^-, \nu\right]\right) \frac{Pr(\nu|\nu^-)Pr(\nu^-)}{Pr(\nu)}$$

Finally, the expression for the expected incremental risk is:

$$E\left[\frac{1}{M^*}\sum_{k=0}^{M^*-1} |\Delta C_k|\right] = \sum_{\nu=[\bar{k},\bar{s}]} \frac{1}{\bar{k}+1}\rho(\nu)Pr(\nu)Pr\{X_{\bar{k}+1} \leq \bar{X}_{\bar{k}+1}|\nu\}$$

5 Computational results in the simplified model

In this section we compare performance characteristics of delta hedging and three local risk minimization hedging strategies for American-type options. We consider the simple model: the option holder can exercise early only at predetermined hedging times, i.e., $n_H = n_E$. We refer to this case as *"the Bermudan option"*[3].

We consider a number of parameter sets and compute local risk minimization hedging strategies as described in the previous section. A binomial tree model is constructed for the asset price assuming $T = 1$, $S_0 = 100$, $\mu = 0.2$, $\sigma = 0.2$ and $r = 0.1$. The number of periods in the binomial tree is $N = 600$.

We compute the holdings (ξ, η) in the portfolio at each node in the binomial tree using, as described in §3, delta-hedging, piecewise local linear risk minimization, local quadratic risk minimization, and constrained local piecewise linear risk minimization.

We compare the performance of these four methods. We generate 100,000 paths for the stock price based on the binomial tree. For each path we determine the moment of early exercise t_{M^*}, and at this moment calculate the final cumulative cost C_{M^*}. Thus, we are able to obtain the distribution of the cumulative costs and, in particular, report mean and 95% quantile of the distribution. We also report mean of the incremental risk.

The results for the Bermudan option with $n_E = n_H$ are presented in Tables 3.1, 3.2, and 5.1-5.6. The calculations are done for different strike prices K and time periods between hedging moments $n_H = N/M$. As can be expected,

[3] We are particularly interested in the situation when $n_H = n_E > 1$ because $n_H = n_E = 1$ corresponds to a standard American option hedged at every time period. However, in the tables we report results for this case as well.

performance differences are small when rebalancing is frequent, with all four methods becoming equivalent when the hedging portfolio is rebalanced at every time period (hence the market is complete).

Table 5.1: Initial Cost of the portfolio for the Bermudan option with $n_E = n_H$

IniCost			n_H						
			1	5	25	50	100	300	600
K	90	L_1	1.7177	0.0333	0.7712	0.0879	0	0	0
		L_2	1.7177	1.7091	1.6678	1.6191	1.5297	1.2517	0.9671
		L_1c	1.7177	1.6531	1.557	1.4867	1.3665	0.9356	0.6516
		Δ	1.7177	1.7146	1.7001	1.6834	1.6518	1.553	1.4254
	100	L_1	4.8149	0.6977	3.2128	1.2822	1.0042	0	0
		L_2	4.8149	4.7994	4.725	4.6353	4.4615	3.8008	2.8703
		L_1c	4.8149	4.7188	4.5696	4.4564	4.2659	3.2412	2.2359
		Δ	4.8149	4.8073	4.7722	4.7295	4.6505	4.3122	3.7499
	110	L_1	10.7182	5.8008	9.4925	7.7425	8.2208	7.2594	2.6164
		L_2	10.7182	10.6976	10.5982	10.4494	10.1128	8.6063	6.4581
		L_1c	10.7182	10.6398	10.4898	10.3542	10.0112	8.4011	5.9606
		Δ	10.7182	10.7034	10.6339	10.5255	10.2811	9.2042	7.7139

The initial hedging cost, C_0, shows differences in hedging style for the strategies considered. Low initial hedging cost might indicate that the hedging strategy requires aggressive financing in the remaining life of the option to match the option payoff at the moment of exercise. Tables 3.1 and 5.1 show that, when hedging is infrequent, delta hedging has significantly higher initial cost than local risk minimization hedging. Note that L_1 local risk minimization has the least initial cost. Comparing to Table 3.1, Table 5.1 shows that the initial costs for Bermudan options vary with the hedging frequency in the same fashion as initial costs for European options except that the initial costs for Bermudan options are higher. Note that the relative difference becomes smaller when hedging less frequently (n_H is large). This is reasonable since the early exercise can occur only at hedging times and thus it is harder for the option holder to benefit from the early exercise opportunity.

Tables 3.2 and 5.2 show that, when rebalancing is infrequent, the average cumulative cost for delta hedging is significantly higher than the average cumulative cost for local risk minimization hedging. In addition, average cumulative costs for Bermudan options change with hedging frequencies in a similar pattern as those for European options. As in the case of European options, the piecewise linear (L_1) method tends to give lower average cumulative cost, especially for out-of-the money options. The difference is most

Table 5.2: Average Cumulative Cost for the Bermudan option with $n_E = n_H$

AvgCumCost			n_H						
			1	5	25	50	100	300	600
K	90	L_1	1.72	1.11	1.42	1.07	0.94	0.68	0.52
		L_2	1.72	1.71	1.67	1.62	1.53	1.25	0.97
		L_1c	1.72	1.65	1.56	1.49	1.37	0.93	0.65
		Δ	1.72	1.72	1.72	1.72	1.73	1.79	1.89
	100	L_1	4.81	4.09	4.53	4.03	3.7	2.41	1.66
		L_2	4.81	4.8	4.72	4.63	4.46	3.8	2.87
		L_1c	4.81	4.72	4.57	4.45	4.26	3.24	2.24
		Δ	4.81	4.81	4.8	4.79	4.78	4.72	4.55
	110	L_1	10.72	10.72	10.65	10.6	10.48	8.53	4.98
		L_2	10.72	10.7	10.59	10.45	10.1	8.62	6.45
		L_1c	10.72	10.64	10.48	10.35	10	8.41	5.96
		Δ	10.72	10.71	10.66	10.58	10.4	9.69	8.73

Table 5.3: 95% quantiles of Cumulative Cost for the European option ($n_E = N$)

q95 CumCost			n_H						
			1	5	25	50	100	300	600
K	90	L_1	1.43	4.33	3.07	4.52	5.21	4.96	3.32
		L_2	1.43	1.86	2.53	3	3.59	4.12	3.6
		L_1c	1.43	1.93	2.64	3.13	3.8	4.41	3.05
		Δ	1.43	1.86	2.53	3.01	3.65	5.3	7.16
	100	L_1	3.75	7.53	6.11	8.08	9.09	12.29	12.37
		L_2	3.75	4.46	5.51	6.21	7.24	9.52	10.23
		L_1c	3.75	4.53	5.64	6.39	7.48	10.19	11.16
		Δ	3.75	4.46	5.52	6.28	7.34	10.55	14.43
	110	L_1	7.71	11.95	10.56	12.77	13.79	15.9	19.33
		L_2	7.71	8.6	9.88	10.74	11.89	15.01	17.24
		L_1c	7.71	8.65	9.95	10.87	12.22	15.35	17.3
		Δ	7.71	8.62	9.96	10.93	12.38	16.69	22.61

Table 5.4: 95% quantiles of Cumulative Cost for the Bermudan option with $n_E = n_H$

q95 CumCost			n_H						
			1	5	25	50	100	300	600
K	90	L_1	1.72	5.15	3.68	5.48	6.16	6.13	3.32
		L_2	1.72	2.24	3.03	3.58	4.2	4.57	3.6
		L_1c	1.72	2.33	3.22	3.83	4.56	5.37	3.05
		Δ	1.72	2.24	3.01	3.54	4.2	5.75	7.16
	100	L_1	4.81	9.82	7.45	9.43	10.87	14.18	12.37
		L_2	4.81	5.61	6.73	7.52	8.58	10.28	10.23
		L_1c	4.81	5.67	6.89	7.75	8.92	11.71	11.16
		Δ	4.81	5.61	6.77	7.54	8.65	11.52	14.43
	110	L_1	10.72	17.65	13.86	17.25	19.4	17.72	19.33
		L_2	10.72	11.51	12.6	13.31	14.51	17.09	17.24
		L_1c	10.72	11.47	12.59	13.46	14.44	16.85	17.3
		Δ	10.72	11.53	12.77	13.69	15.09	18.77	22.61

Table 5.5: Average Incremental Risk for the European option

MeanIncRisk			n_H						
			1	5	25	50	100	300	600
K	90	L_1	0.00	0.01	0.05	0.09	0.17	0.36	0.51
		L_2	0.00	0.01	0.07	0.13	0.25	0.68	1.12
		L_1c	0.00	0.01	0.06	0.13	0.25	0.61	0.90
		Δ	0.00	0.01	0.07	0.14	0.29	0.97	2.21
	100	L_1	0.00	0.02	0.11	0.23	0.45	1.23	1.63
		L_2	0.00	0.02	0.12	0.24	0.48	1.42	2.60
		L_1c	0.00	0.02	0.12	0.24	0.48	1.40	2.37
		Δ	0.00	0.02	0.12	0.25	0.51	1.76	4.06
	110	L_1	0.00	0.04	0.15	0.35	0.70	2.18	3.79
		L_2	0.00	0.03	0.16	0.32	0.67	2.11	4.21
		L_1c	0.00	0.03	0.16	0.32	0.65	2.15	4.13
		Δ	0.00	0.03	0.16	0.33	0.69	2.36	5.47

Table 5.6: Average Incremental Risk for the Bermudan option ($n_E = n_H$)

MeanIncRisk			n_H						
			1	5	25	50	100	300	600
K	90	L_1	0.00	0.01	0.07	0.13	0.27	0.51	0.51
		L_2	0.00	0.02	0.08	0.16	0.33	0.81	1.12
		L_1c	0.00	0.02	0.08	0.17	0.32	0.74	0.90
		Δ	0.00	0.02	0.08	0.17	0.36	1.11	2.21
	100	L_1	0.00	0.05	0.20	0.63	1.17	2.11	1.63
		L_2	0.00	0.04	0.18	0.39	0.76	1.81	2.60
		L_1c	0.00	0.04	0.19	0.41	0.79	1.88	2.37
		Δ	0.00	0.04	0.18	0.38	0.75	2.08	4.06
	110	L_1	0.00	0.23	0.41	0.84	0.96	2.77	3.79
		L_2	0.00	0.07	0.33	0.59	1.05	2.61	4.21
		L_1c	0.00	0.07	0.39	0.61	1.06	2.63	4.13
		Δ	0.00	0.06	0.31	0.55	0.97	2.69	5.47

striking when the market is more incomplete (e.g., hedging is infrequent in the Black-Scholes setting). Similar to the case of European options, performance of constrained piecewise linear (L_1c) method is between L_1 and L_2.

Tables 5.3 and 5.4 report 95% quantiles of cumulative costs. Although delta hedging incurs higher average cumulative costs, we observe that the tail risk VaR is much higher than those of local risk minimization methods when rebalancing is highly infrequent. In addition, the 95% quantiles are higher for L_1 than for L_2 which is consistent with the conclusion in [3] that L_1 method tends to produce larger right tails for the distribution of cumulative costs. However, for the Bermudan option, though the same pattern holds, the difference is sometimes larger than in the case of European option. This can be explained by the following argument. Since the L_1 objective function does not penalize large deviations as much as L_2 does, larger incremental cost may happen with a significant probability. But the Bermudan option, which is more beneficial for the option holder due to the early exercise feature, has more opportunities for large losses for the writer, thus boosting the right tail of cumulative cost distribution.

In order to underscore this difference, we compare distributions of the cumulative costs for the European and Bermudan options. Figure 2 present histograms corresponding to L_1 risk minimization and Figure 3) present distributions corresponding to L_2 risk minimization. We see that, for L_2, the distributions are close to normal in both cases and differ only in the mean and variance. For L_1, however, while the distribution in the case of the European

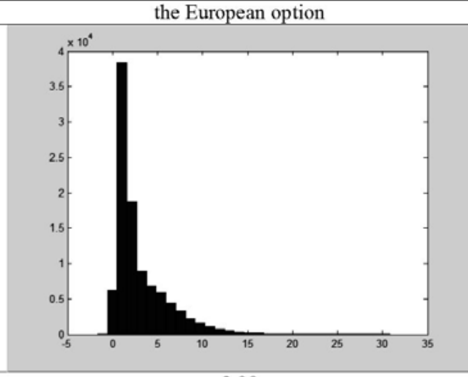
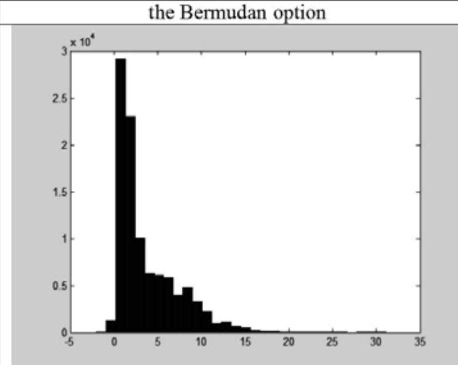
<table>
<tr><td></td><td>the European option</td><td>the Bermudan option</td></tr>
<tr><td>L1</td><td>
</td><td>
</td></tr>
<tr><td>Mean</td><td>2.99</td><td>3.69</td></tr>
<tr><td>StdDev</td><td>2.98</td><td>3.46</td></tr>
</table>

Figure 2: Cumulative cost distributions for European option ($n_E = N$) and the Bermudan option ($n_E = n_H$); L_1 optimization is used. Calculations are performed for parameter values $K = 100, M = 6$.

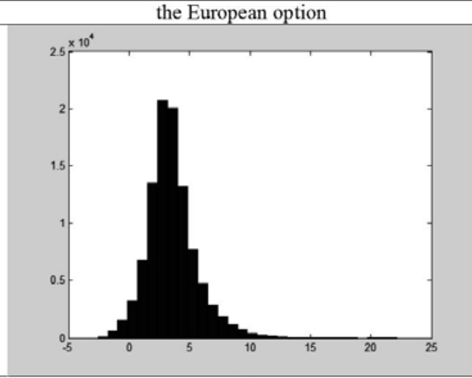
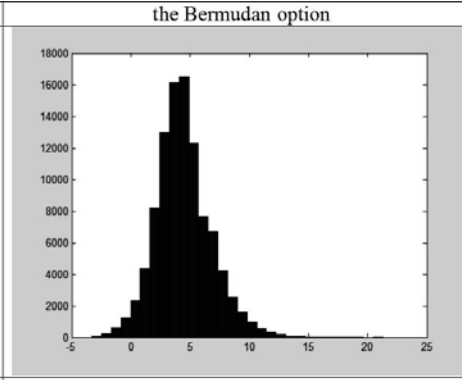
<table>
<tr><td></td><td>the European option</td><td>the Bermudan option</td></tr>
<tr><td>L2</td><td>
</td><td>
</td></tr>
<tr><td>Mean</td><td>3.56</td><td>4.45</td></tr>
<tr><td>StdDev</td><td>2.05</td><td>2.34</td></tr>
</table>

Figure 3: Cumulative cost distributions for European option ($n_E = N$) and the Bermudan option ($n_E = n_H$); L_2 optimization is used. Calculations are performed for parameter values $K = 100, M = 6$.

option has a nicely declining tail, the right tail of the cumulative distribution for the Bermudan option is more uneven.

Figure 4 contrasts empirical cumulative distribution functions (CDFs) for cumulative costs obtained by different methods. We observe that L_1 method produces a significantly different shape for the CDF. In particular, probabilities of small cumulative costs are significantly higher than the corresponding probabilities for the other three methods.

The average incremental risk is lower for L_1 methods than for the L_2 for out-of-the money puts - the effect also observed in the European option case. However, while this is true in the case of European options for both out-of-the money and at-the-money ($K = 100$) options, for the Bermudan option this holds only when the option is out-of-the-money.

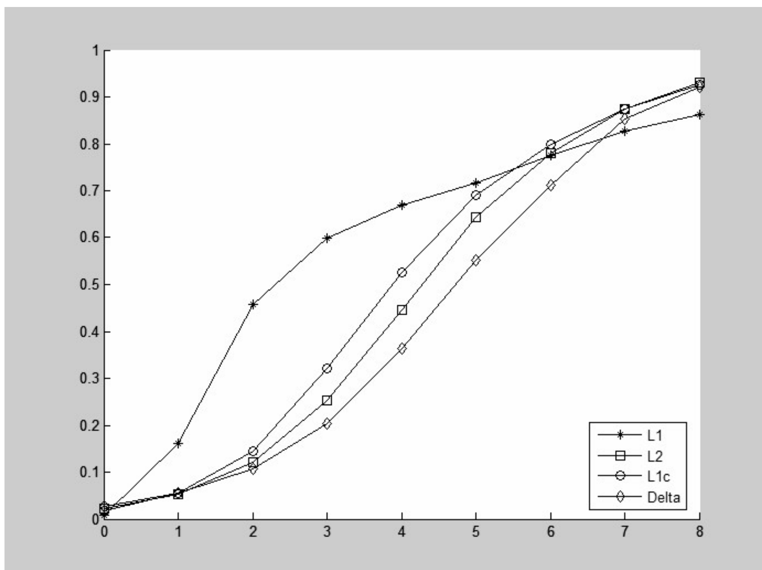


Figure 4: Cumulative cost CDFs for the Bermudan option ($n_E = n_H$); Calculations are performed for parameter values $K = 100, M = 6$.

6 A general model: hedging times different from exercising times

To this point we have considered the case when the option holder is permitted to exercise only at predetermined hedging times. However, a more practical situation is when the early exercise can occur at any moment ($n_E = 1$) (or at least more often than when the hedging portfolio is rebalanced). Recall that we refer to this case as *the American option.*

We can extend the local risk minimization technique to this framework. Suppose that we are formulating the objective function at hedging node $[k, j]$ in order to determine the portfolio holdings. For each subtree, we need to calculate the probabilities of reaching terminal points and calculate payoffs or continuation values at these points and use them, along with the probability of arriving at a given terminal point, to formulate a corresponding term in the objective function, see Fig. 5.

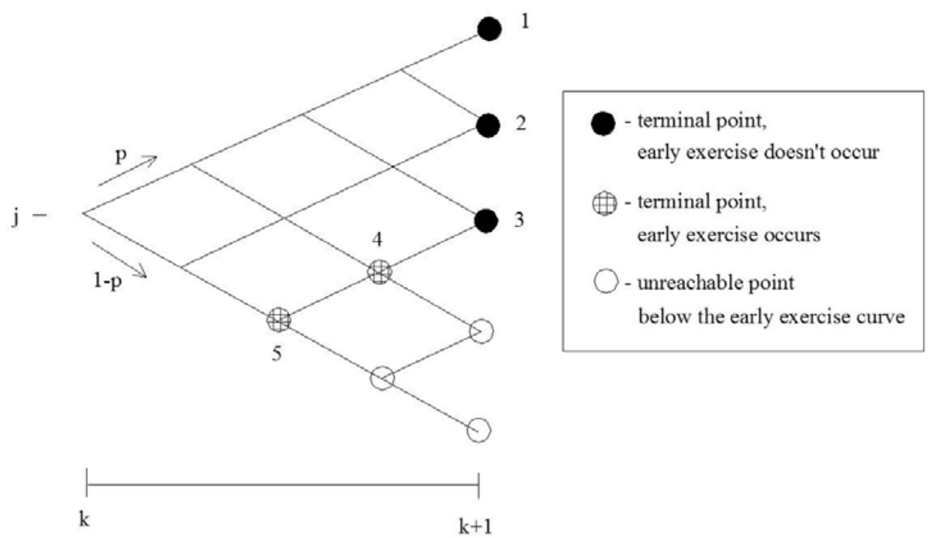


Figure 5: Illustration of the algorithm in case the early exercise can occur at any moment.

Earlier, when we assumed that the early exercise can occur only at hedging times, subtrees for all points were the same. Now, subtrees can be different. Processing of subtrees makes the problem computationally more intensive.

However, we can speed up calculations by observing that if all the lowest nodes of the subtree are above the early exercise critical values then no early exercise can occur in the whole subtree and the objective function can be constructed in the usual way. If the initial point $[k, j]$ lies below the early exercise critical value then there is no need to calculate hedge positions since the hedging portfolio should have been liquidated at that point.

All other definitions from the simplified model are carried through with minor adjustments, for example, M^* is defined as the number of the subtree in which the early exercise occurs.

To allow easy comparisons, numerical results are presented for the same set of parameters as in the simplified model. We observe that average cumulative costs are lower for L_1 methods than for the L_2 when the option is out-of-the-money. However, behavior of the average incremental risk is not as clear as in the simplified model.

Moreover, if we compare the average cumulative cost and average incremental risk for the simplified and general models (Tables 5.2, 6.2, 5.6, and 6.4) we observe that, in the general case, average cumulative cost is higher while average incremental risk becomes lower. A possible explanation is that each subtree in the general model (see Fig. 5) is 'narrower'; hence there is less variance in the incremental cost and the local optimization problem can be solved more successfully.

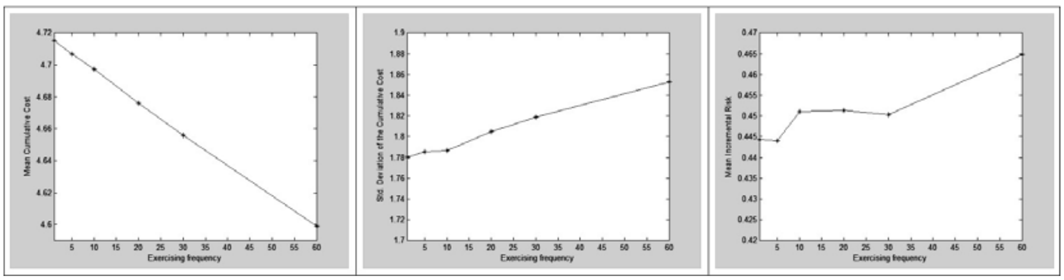


Figure 6: Dependence of cumulative cost and incremental risk on the exercising frequency

In fact, we can show that this holds for intermediate values of n_E. For $K = 100, n_H = 60$ we plot the average and the standard deviation of the cumulative cost and the average incremental risk for L_2 strategy as n_E changes from 1 (the American option) to n_H (the Bermudan option)[4] (see Figure 6). We see that the average cumulative cost declines as n_E increases. The standard deviation of the cumulative cost, on the other hand, gradually goes up. The behavior of the mean incremental risk is trickier - it grows in steps.

So far we have considered put options. We note that the risk minimization framework and solution methods can readily be applied to call options as well as other more complex options.

In [3] the discrete hedging was done for the European put option and then the discrete hedging put-call parity was established, providing a way to calculate the portfolio holdings for the European call option.

In the case of American options, however, there is no put-call parity. For standard American and Bermudan call options it is never optimal to exercise the option early if there is no dividend payment; the solution coincides with the one for the European call option.

[4]we do the calculations only when n_E is a divisor of n_H.

Table 6.1: Initial Cost of the portfolio for the American option ($n_E = 1$)

IniCost			n_H						
			1	5	25	50	100	300	600
K	90	L_1	1.7177	0.0350	0.7793	0.1253	0.0588	0.0000	0.0000
		L_2	1.7177	1.7122	1.6864	1.6573	1.6072	1.4752	1.3894
		L_1c	1.7177	1.6571	1.5926	1.5529	1.4838	1.2115	1.0548
		Δ	1.7177	1.7177	1.7177	1.7177	1.7177	1.7177	1.7177
	100	L_1	4.8149	0.7379	3.2278	1.5294	1.8541	2.0861	0.0000
		L_2	4.8149	4.8070	4.7703	4.7298	4.6632	4.5158	4.4367
		L_1c	4.8149	4.7285	4.6591	4.6260	4.5741	4.3703	4.3318
		Δ	4.8149	4.8149	4.8149	4.8149	4.8149	4.8149	4.8149
	110	L_1	10.7182	5.9696	9.5724	9.1121	9.8727	11.4220	12.4540
		L_2	10.7182	10.7127	10.6897	10.6678	10.6367	10.5809	10.5613
		L_1c	10.7182	10.6592	10.6740	10.7093	10.7379	10.9415	11.0943
		Δ	10.7182	10.7182	10.7182	10.7182	10.7182	10.7182	10.7182

Table 6.2: Average Cumulative Cost for the American option ($n_E = 1$)

MeanCumCost			n_H						
			1	5	25	50	100	300	600
K	90	L_1	1.72	1.12	1.43	1.15	1.08	0.88	0.81
		L_2	1.72	1.71	1.69	1.66	1.61	1.47	1.38
		L_1c	1.72	1.66	1.59	1.55	1.48	1.21	1.05
		Δ	1.72	1.72	1.74	1.76	1.82	2.07	2.50
	100	L_1	4.81	4.13	4.58	4.26	4.23	3.85	3.05
		L_2	4.81	4.81	4.77	4.73	4.66	4.52	4.43
		L_1c	4.81	4.73	4.66	4.62	4.57	4.37	4.32
		Δ	4.81	4.82	4.86	4.91	5.01	5.58	6.61
	110	L_1	10.72	10.79	10.76	10.87	10.85	10.79	10.91
		L_2	10.72	10.72	10.69	10.66	10.63	10.57	10.55
		L_1c	10.72	10.66	10.68	10.70	10.73	10.93	11.09
		Δ	10.72	10.73	10.77	10.83	11.00	11.74	12.98

Table 6.3: 95% quantiles of the Cumulative Cost for the American option ($n_E = 1$)

q95 CumCost			n_H						
			1	5	25	50	100	300	600
K	90	L_1	1.72	5.17	3.69	5.32	6.10	7.79	9.11
		L_2	1.72	2.24	3.04	3.63	4.39	6.11	7.69
		L_1c	1.72	2.34	3.23	3.86	4.70	6.67	8.52
		Δ	1.72	2.24	3.03	3.59	4.37	6.65	8.27
	100	L_1	4.81	10.01	7.46	9.63	9.88	10.80	12.71
		L_2	4.81	5.61	6.72	7.44	8.57	9.88	10.59
		L_1c	4.81	5.68	6.96	7.78	8.81	10.12	10.73
		Δ	4.81	5.62	6.80	7.69	8.65	12.64	19.34
	110	L_1	10.72	18.23	13.90	17.09	17.03	16.91	16.45
		L_2	10.72	11.51	12.45	12.97	13.45	13.47	13.41
		L_1c	10.72	11.49	12.87	13.80	14.71	17.57	18.00
		Δ	10.72	11.55	12.81	13.89	15.76	22.73	34.87

Table 6.4: Average Incremental Risk for the American option ($n_E = 1$)

MeanIncRisk			n_H						
			1	5	25	50	100	300	600
K	90	L_1	0.00	0.01	0.07	0.14	0.29	0.66	0.81
		L_2	0.00	0.02	0.08	0.17	0.33	0.90	1.52
		L_1c	0.00	0.02	0.08	0.17	0.33	0.86	1.35
		Δ	0.00	0.02	0.08	0.17	0.36	1.21	2.68
	100	L_1	0.00	0.05	0.20	0.54	1.03	2.37	3.05
		L_2	0.00	0.03	0.18	0.37	0.75	1.98	3.27
		L_1c	0.00	0.03	0.18	0.37	0.74	2.00	3.26
		Δ	0.00	0.03	0.17	0.36	0.72	2.11	4.81
	110	L_1	0.00	0.21	0.42	0.62	0.86	1.65	2.89
		L_2	0.00	0.06	0.32	0.54	0.90	1.97	3.38
		L_1c	0.00	0.07	0.26	0.40	0.68	1.64	3.13
		Δ	0.00	0.06	0.26	0.41	0.67	1.98	5.12

7 Conclusions

When a market is incomplete and the asset price model is complex, hedging American-type options is more difficult than hedging European options in theory as well as in practice. In these situations total risk minimization is computationally more expensive. In practice, simpler approaches like delta-hedging are frequently used. However, using overly simplified methods may result in poor hedging strategy performance when the market is significantly incomplete. Hence identifying methods that are computationally attractive and sufficiently sophisticated to produce good hedging strategies is of both theoretical and practical importance. Local risk minimization methods are examples of such methods.

In this paper we have evaluated performances of delta-hedging, quadratic, and piecewise linear local risk minimization methods for discrete hedging of American-type options. Local risk minimization methods are relatively easy to implement and outperform delta-hedging when the market is highly incomplete. Specifically, delta-hedging can incur higher cost with lesser risk reduction when the market is sufficiently incomplete. In addition, when hedging rebalancing is infrequent, performance of various local risk minimization methods can be significantly different. In particular, hedging performance depends on moneyness of the option. For example, the piecewise linear local risk minimization method and its modifications tend to perform better than quadratic risk minimization when the option is out-of-the-money.

We have observed that local risk minimization methods do not necessarily achieve optimality in terms of global risk measures, e.g., average incremental cost per period. This suggests that these methods may suffer from locality in this rather complex framework. Hence it would be interesting to investigate how close local risk minimization methods can approach the optimal total risk measures achievable by more complex total risk minimization methods.

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