Extensions of the cycle matroid of a complete graph

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Q: What is a matroid?

→ generalizes the concept of independence (e.g. linear independence)

→ introduced by Whitney and (independently) by Nakasawa in 1935.

Representable Matroids

$$A = \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Let $E = \{e_1, e_2, e_3, e_4\}$ be the set of columns of A.

Let $\mathcal{I} = \{ I \subseteq E : I$ is linearly independent over $GF(2).\}$

$= \{\phi$, all 1-subsets, all 2-subsets, and all 3-subsets other than $\{e_1, e_3, e_4\}\}$

$M(A) = (E, \mathcal{I})$ is an example of a matroid.

Properties of $\mathcal{I}$:

(1) $\emptyset \in \mathcal{I}$

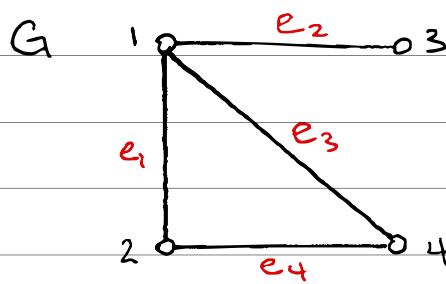
(2) If $I \in \mathcal{I}$, then $J \in \mathcal{I}$ for all $J \subseteq I$.

(3) If $I, J \in \mathcal{I}$ and $|I| = |J| + 1$, then $\exists e \in I$ s.t. $J \cup \{e\}$ is in $\mathcal{I}$.

A matroid M is a pair $(E, \mathcal{I})$ where E is a finite set and $\mathcal{I}$ is a collection of subsets of E that satisfies (1), (2), and (3).

E is called the ground set of M.
The subsets in $\mathcal{I}$ are called independent sets.

Graphic Matroids



Let $E = \{e_1, e_2, e_3, e_4\}$ be the edge set of G.

Let $\mathcal{I} = \{ I \subseteq E : I \text{ acyclic} \}$

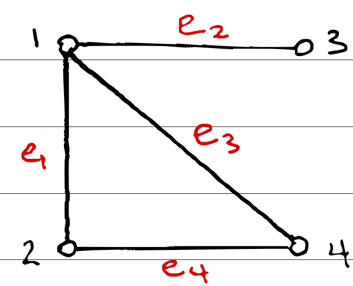
contains all forests of G

Exercise: Use graph theory to prove $\mathcal{I}$ satisfies (1), (2), and (3).

$M(G) = (E, \mathcal{I})$ is another example of a matroid.

"cycle" or "graphic" matroid.

G



$$A_G = \begin{array}{c|cccc} & e_1 & e_2 & e_3 & e_4 \\ \hline 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 \\ 3 & 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 1 & 1 \end{array}$$

Consider the incidence matrix of G.

forests in G $\longleftrightarrow$ linear independent sets (over GF(2)) in A_G

$M(G) = M(A_G)$.

For $E' \subseteq E$, rank(E') = size of a largest independent set in E'.

\# vertices in G − \# components in (V, E')

Motivation

Q: How many matroids on n elements are there? $m(n)$

$$2^{\frac{1}{n}\binom{n}{n/2}} \leq m(n) \leq 2^{\frac{2}{n}\binom{n}{n/2}(1+o(1))}$$

Graham and Sloane ('80) and Knuth ('74)

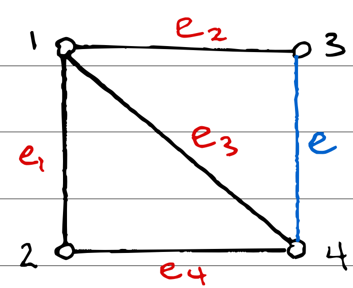
Bansal, Pendavingh, Van der Pol ('15)

→ $m(n)$ is doubly exponential

→ $m(n)$ is "close" to 2^{2^n}.

Extensions

G'



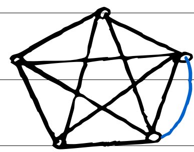
$$A_{G'} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$M(G')$ is an extension of $M(G)$.

In general, M is an extension of N if $M \backslash e = N$ for some $e \in M$.

Q: How many extensions of a matroid are there?

Q: How many extensions of $M(K_{r+1})$ are there?



$$\begin{array}{c c} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 1 \end{bmatrix} & \begin{array}{c} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{array} \end{array}$$

Theorem (Nelson, R., Van der Pol, '22):

$$\#\text{extensions of } M(K_{r+1}) = 2^{\binom{r}{r/2}(1+o(1))}.$$

$M(K_{\sqrt{2n}})$ has $\approx n$ elements.

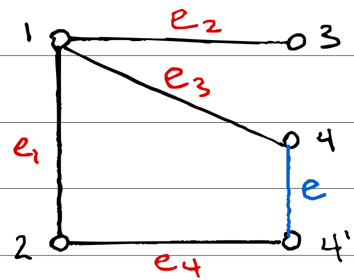
$$\#\text{exts of } M(K_{\sqrt{2n}}) \approx 2^{\binom{\sqrt{2n}}{\sqrt{2n}/2}} \approx 2^{\frac{1}{\text{poly}(n)} 2^{\sqrt{2n}}}$$

doubly exponential
in the # of elements

Recall: $m(n)$ is "close" to 2^{2^n}.

Coextensions

G'



$$A_{G'} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 4' \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

$M(G')$ is a coextension of $M(G)$.

In general, M is a coextension of N if $M/e = N$ for some $e \in M$.

Theorem (Nelson, Van der Pol, '19):
$\#$ coexts of $M(K_{r+1}) = 2^{\frac{1}{2}r!(1+o(1))}$.

matroids on n elements	2^{2^n}
extensions of $M(K_{\sqrt{2n}})$	$2^{2^{\sqrt{2n}}}$
coextensions of $M(K_{\sqrt{2n}})$	$2^{2^{\sqrt{2n}\log\sqrt{2n}}}$
extensions of $PG(\log n, 2)$ projective geometry with $\approx n$ elements	$2^{\log^2 n}$
representable matroids on $[n]$	$\leq 2^{n^3}$ (Nelson, '18)

Q: Are all extensions of $M(K_{r+1})$ representable?

$\rightarrow$ No. $\quad 2^{2^{\sqrt{2n}}} \gg 2^{n^3}$

Q: How do we find all extensions?

(Crapo, '65)

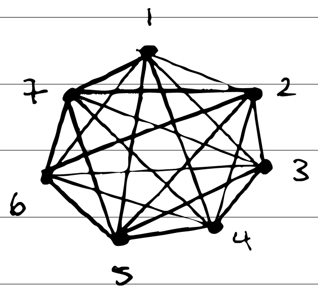
extensions of M $\Longleftrightarrow$ linear subclasses of M

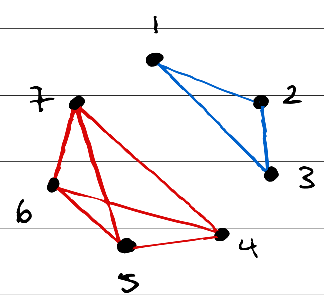
a set of hyperplanes with a certain property.

Hyperplanes are maximal non-spanning sets.

Q: What is a hyperplane in $M(K_{r+1})$?

- rank $r-1$
- adding an edge increases the rank.





hyperplanes

$\Longleftrightarrow$

bipartitions of $[r+1]$

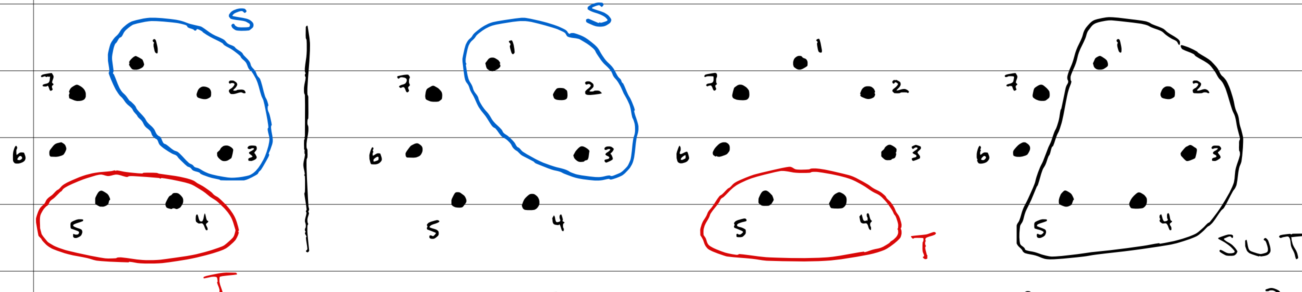
$\Longleftrightarrow$

subsets of $[r]$.

The tripartition property

A set $\mathcal{H}$ of hyperplanes (subsets of $[r]$) has the **tripartition property** if

$\forall$ nonempty disjoint $S, T \subseteq [r]$ either 0, 1, or 3 of $S, T, S \cup T$ are in $\mathcal{H}$.

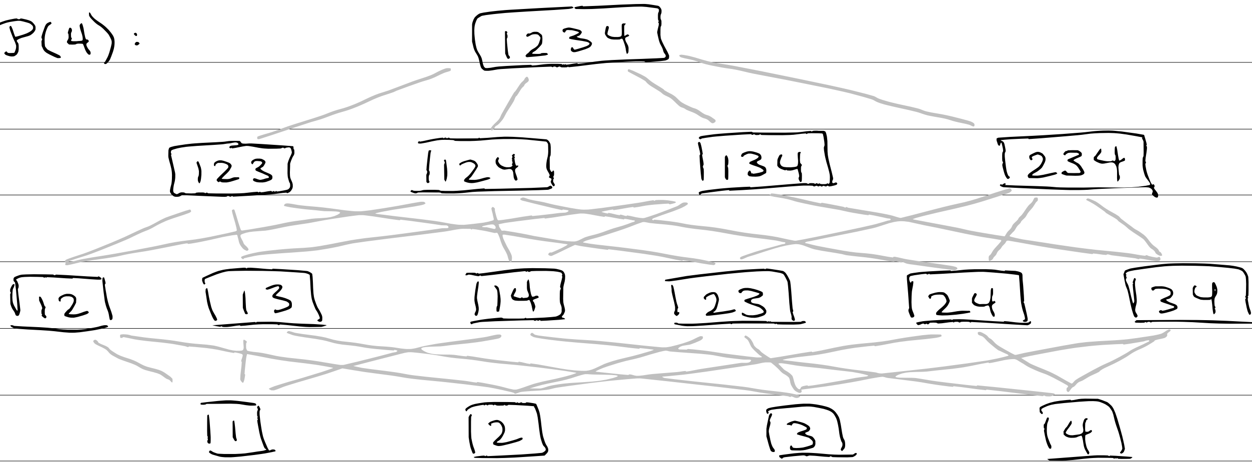


$\{1, 2, 3\}$ $\quad$ $\{4, 5\}$ $\quad$ $\{1, 2, 3, 4, 5\}$

each pair is either comparable or disjoint.

Proof Idea: Boolean lattice $P(r)$

$P(4)$:



subsets of $[r]$ where two are related if they are comparable (ie one is a subset of the other)

Antichain: collection of pairwise non-comparable sets.

Intersecting antichain: collection of sets that are pairwise not comparable or disjoint.

linear subclasses with ≤ 1 from each triple $\{S, T, S \cup T\}$ $\iff$ intersecting antichains of $\mathcal{P}(r)$

The largest intersecting antichain (row $\lceil r+1/2 \rceil$) has size $\alpha = \binom{r}{\lceil r+1/2 \rceil} = \binom{r}{r/2}(1+o(1))$.
Since every subset of an intersecting antichain is an intersecting antichain, we get the following:

Lower Bound:

$$\#\text{ intersecting antichains} \ge 2^{\alpha} = 2^{\binom{r}{r/2}(1+o(1))}.$$

Theorem (Kleitman, '69):

$$\#\text{ antichains} = 2^{\binom{r}{r/2}(1+o(1))}.$$

Upper Bound:

$$\#\text{ intersecting antichains} \le \#\text{ antichains} = 2^{\binom{r}{r/2}(1+o(1))}.$$

$\vdots$

⟨ a bit more work to show the # of all linear subclasses is not "much" more than the # of linear subclasses with ≤ 1 from each triple $\{S, T, S \cup T\}$. ⟩

In the end,

$$\#\text{ extensions} = \#\text{ linear subclasses} = 2^{\binom{r}{r/2}(1+o(1))}.$$