Counting Antichains in the Boolean Lattice

Shayla Redlin

June 10, 2021

PMath and C&O Joint Colloquium

The Container Method

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- 2005: General container method for graphs developed by Sapozhenko.
- 2010's: Balogh, Morris, and Samotij and Saxon and Thomason (indep'ly) developed a general container method for hypergraphs.

The Container Method

Many recent applications:

The Container Method

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- antichains in the Boolean lattice

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- Sidon sets

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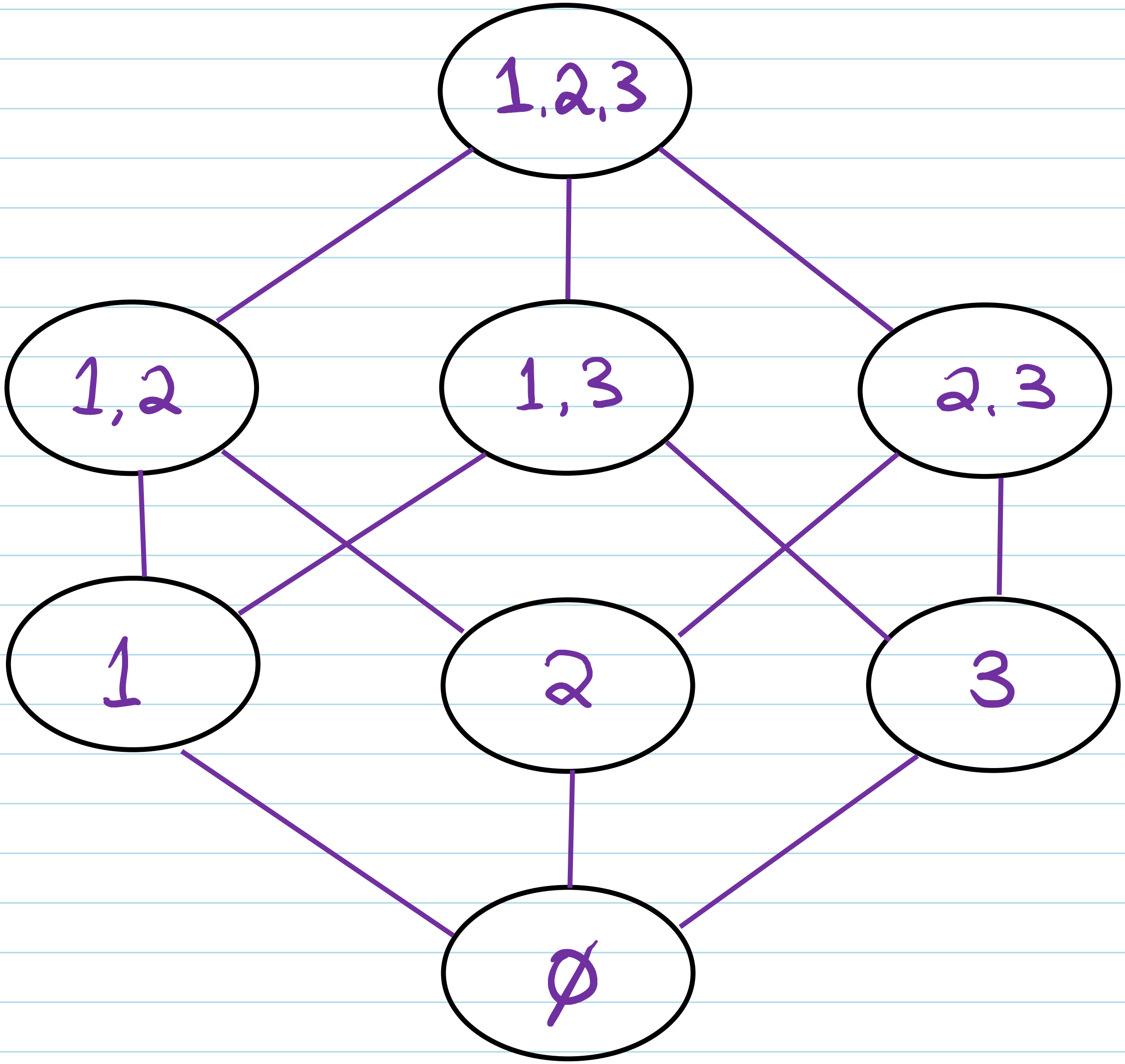
The Container Method

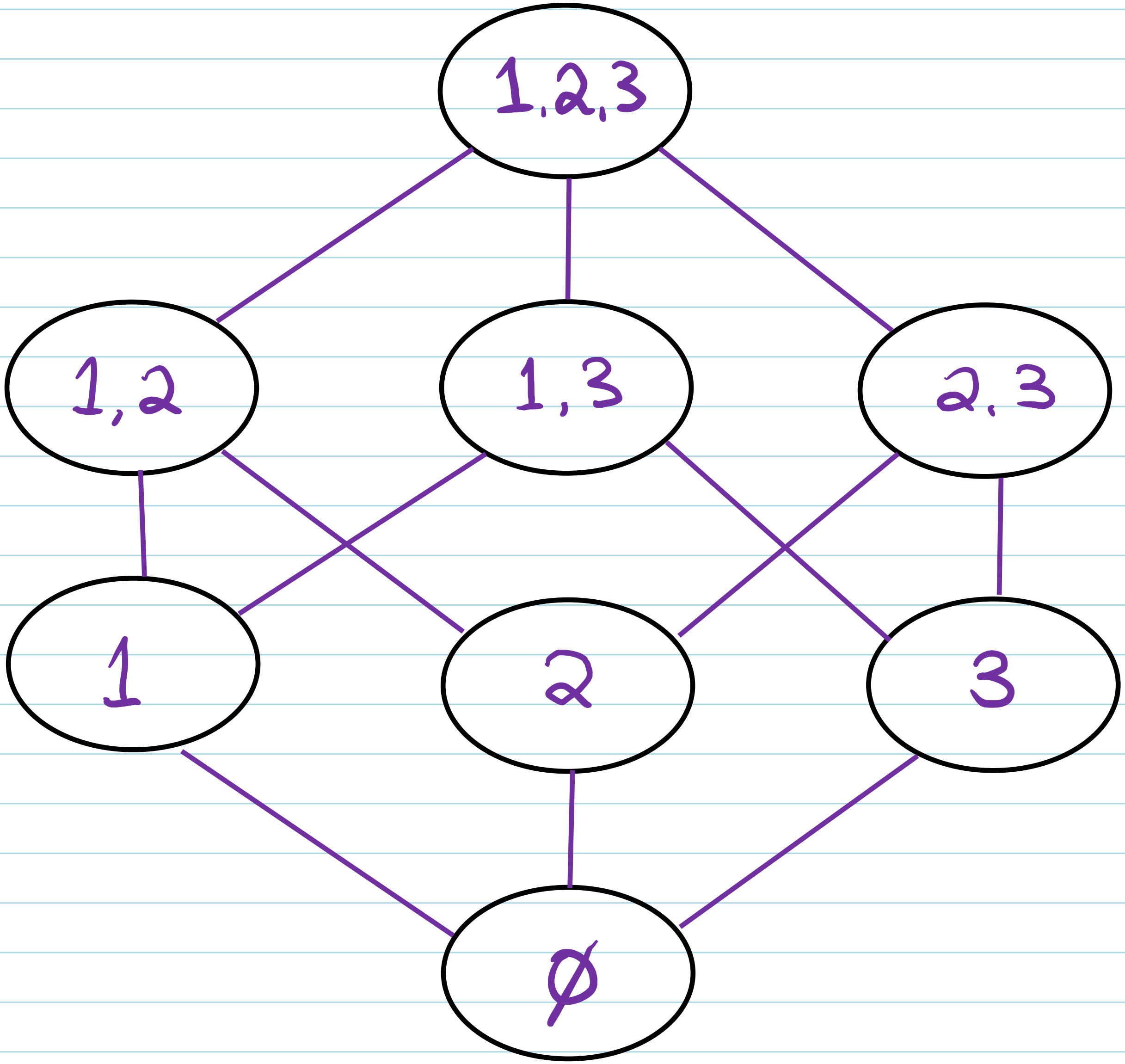
Many recent applications:

- antichains in the Boolean lattice
- Sidon sets
- graph colourings
- triangle-free graphs
- H-free graphs
- and many more...

Boolean Lattice $\mathcal{P}(n)$



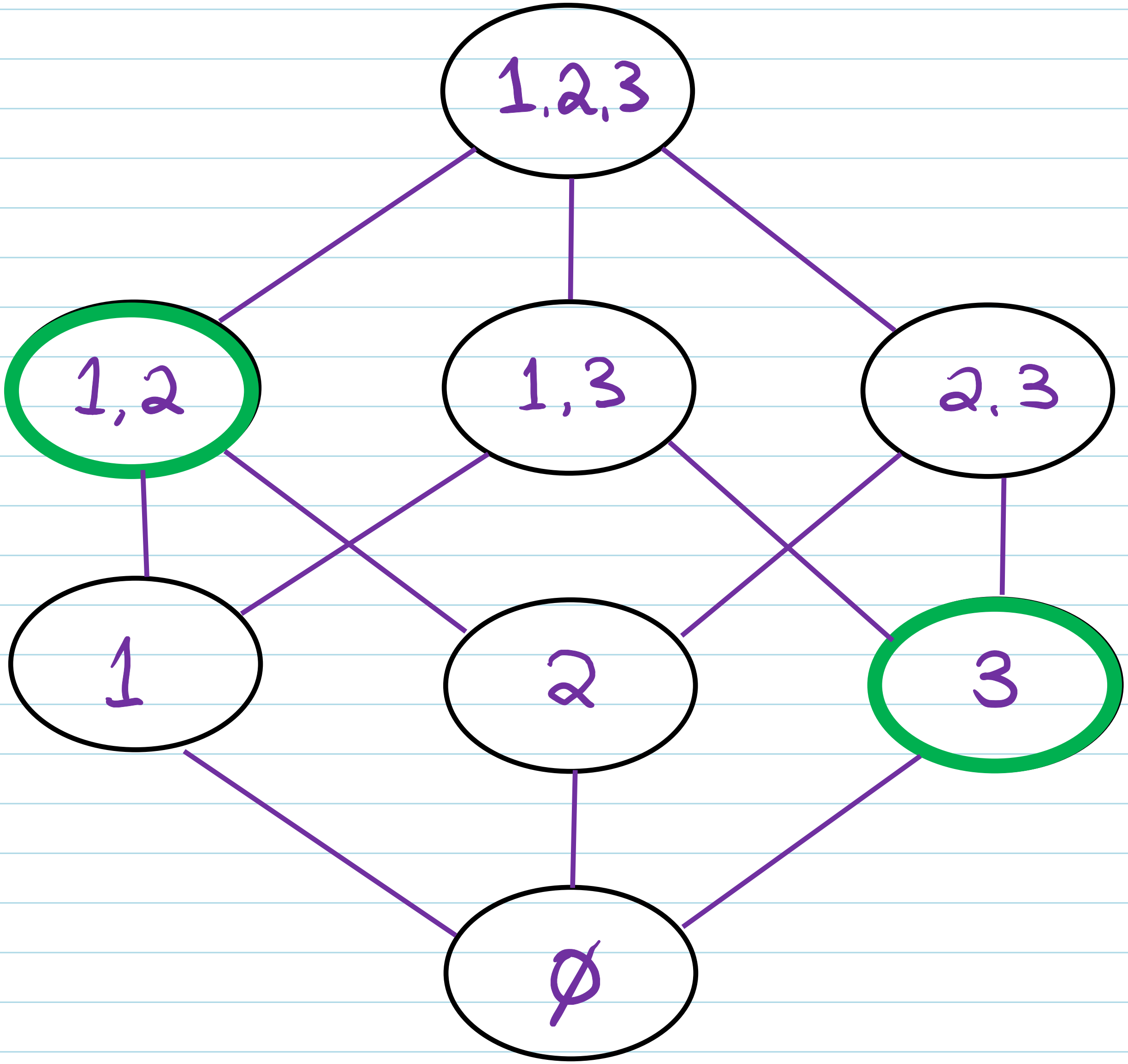
Boolean Lattice $P(n)$



 # elements:

2^n

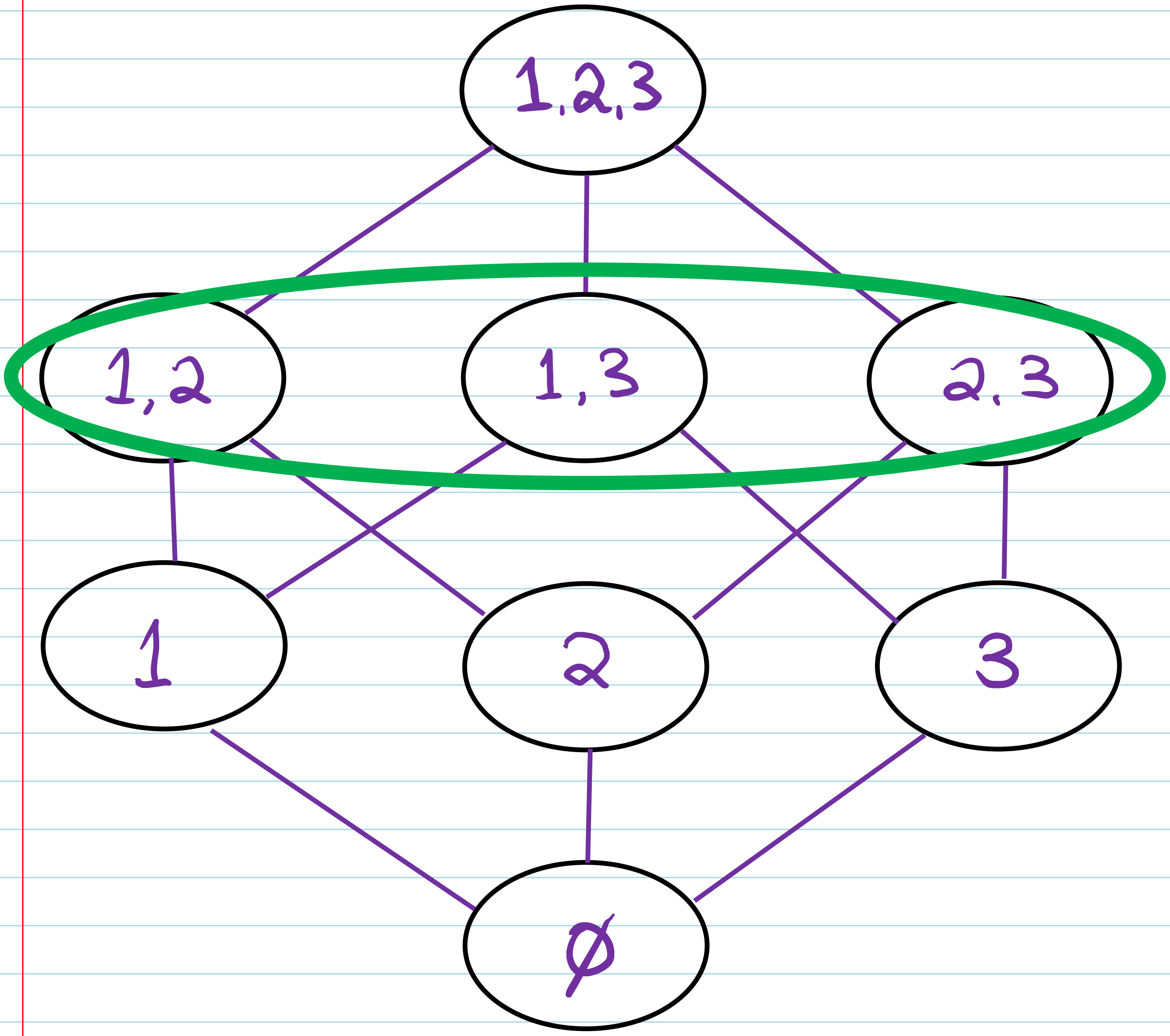
Boolean Lattice $\mathcal{P}(n)$



Antichain

no pair of elements are comparable

Boolean Lattice $\mathcal{P}(n)$



Antichain

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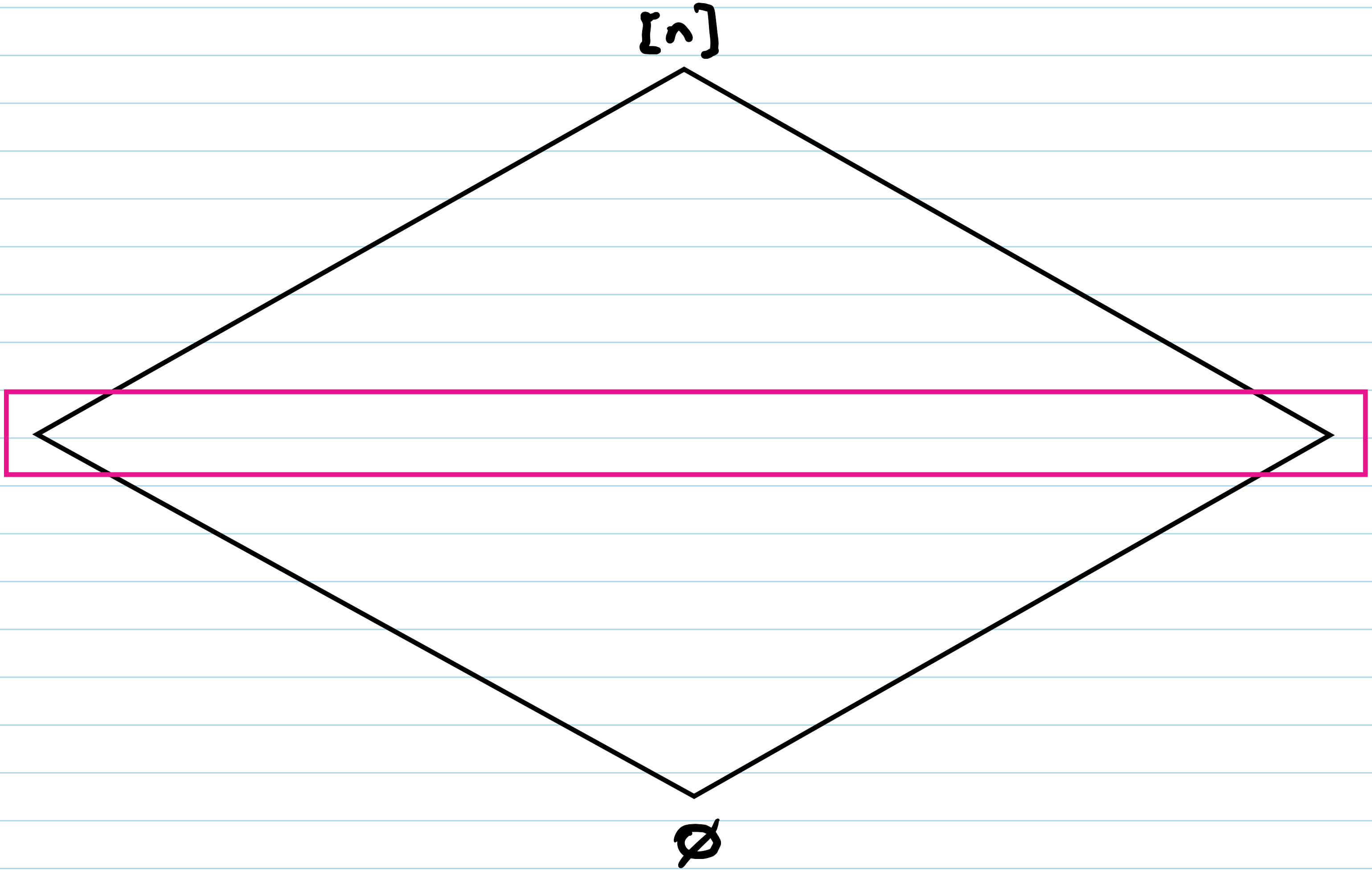
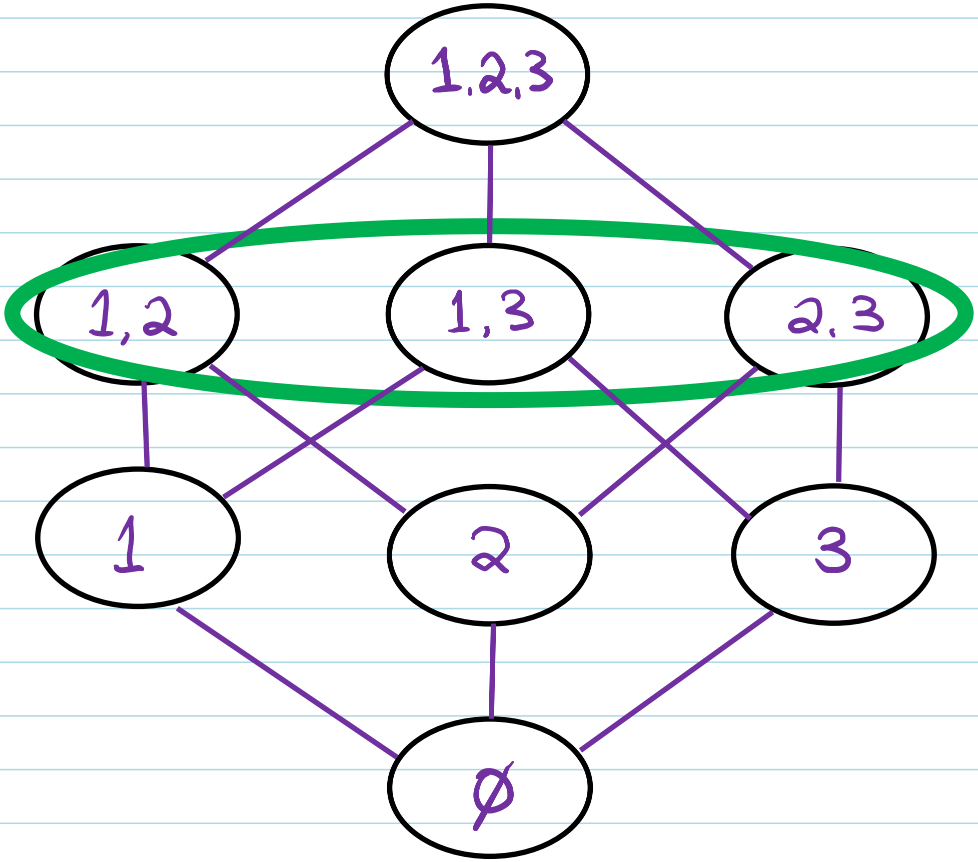
Boolean Lattice $\mathcal{P}(n)$

Theorem (Sperner, 1928):
Largest antichain has size $\binom{n}{n/2}$.



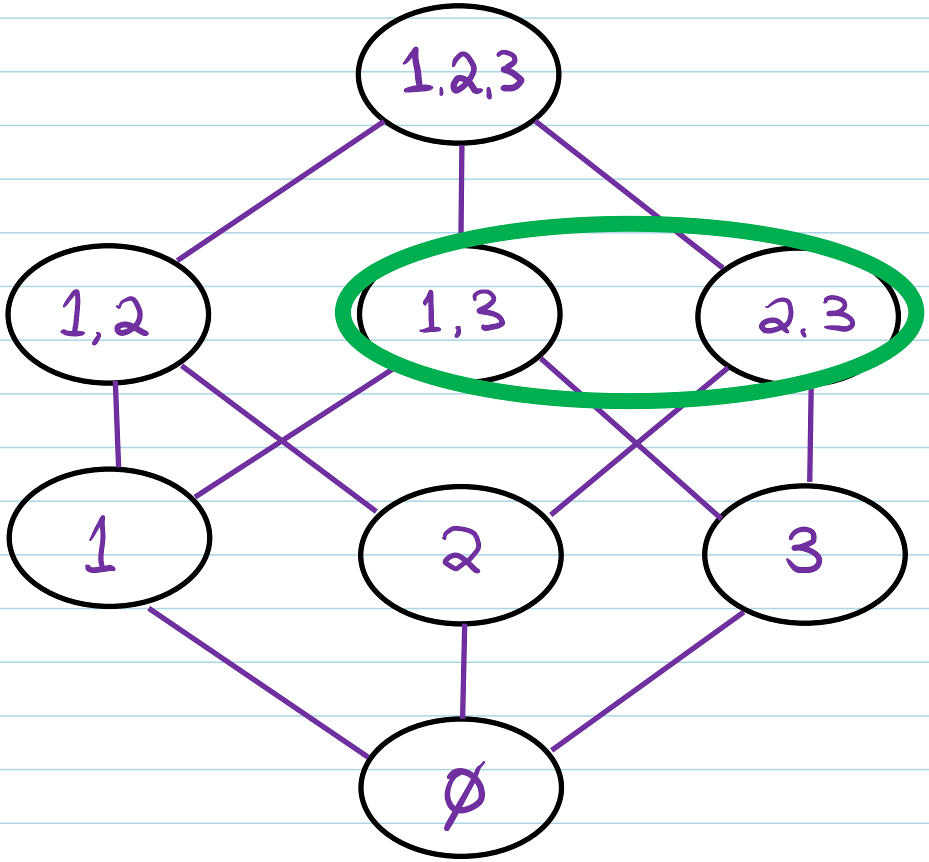
Natural Lower Bound



Observe:

Every subset of an antichain is an antichain.

Natural Lower Bound



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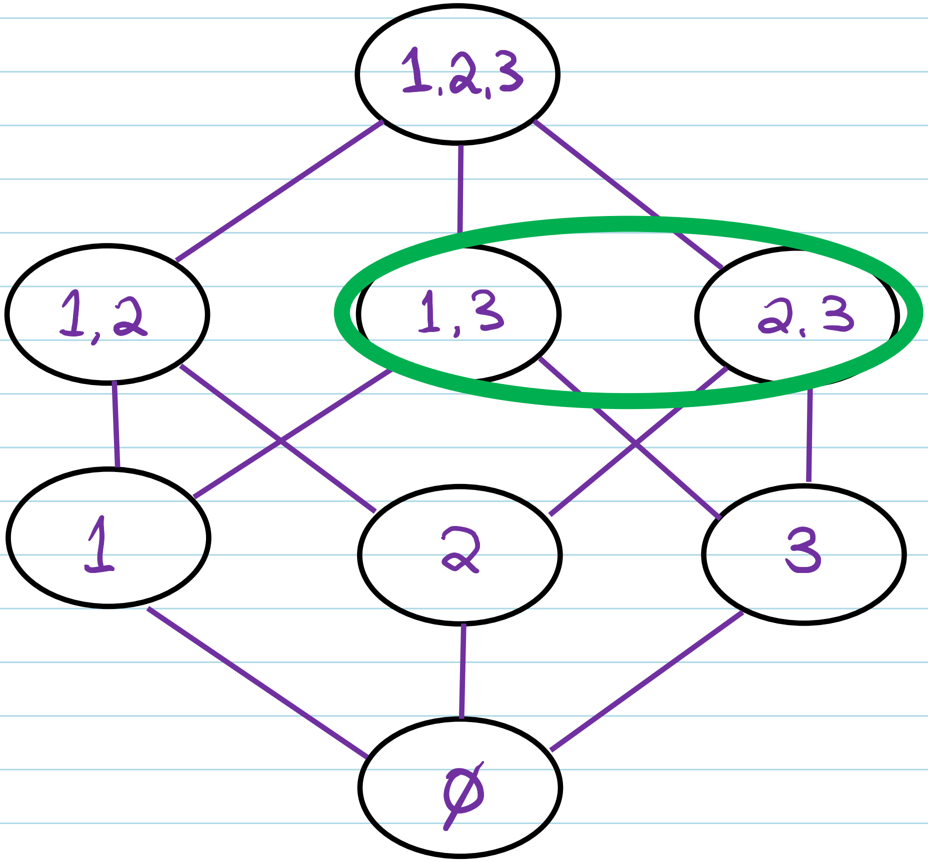

Observe:

Every subset of an antichain is an antichain.

$$\# \text{antichains} \geq \overbrace{2 \cdot 2 \cdot \ldots \cdot 2}^{\binom{n}{n/2}}$$

$$= 2^{\binom{n}{n/2}}$$

A Surprising Upper Bound

Theorem (Kleitman, 1969):

$$\#\text{antichains} \leq 2^{\binom{n}{n/2}(1+o(1))}$$

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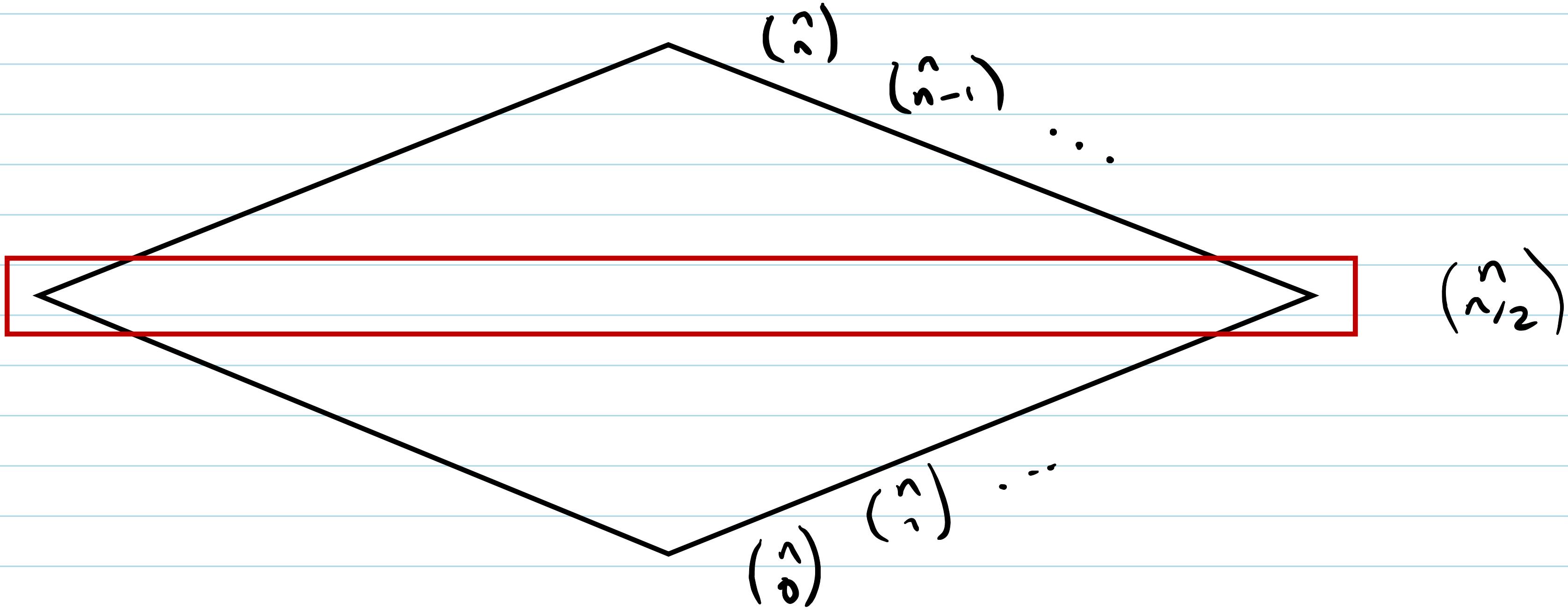

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Proof Idea (Balogh, Treglown, Wagner, 2013):

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- Bound the # of containers.

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Proof Idea (Balogh, Treglown, Wagner, 2013):

- Create a collection of sets where each antichain is *contained* in one.
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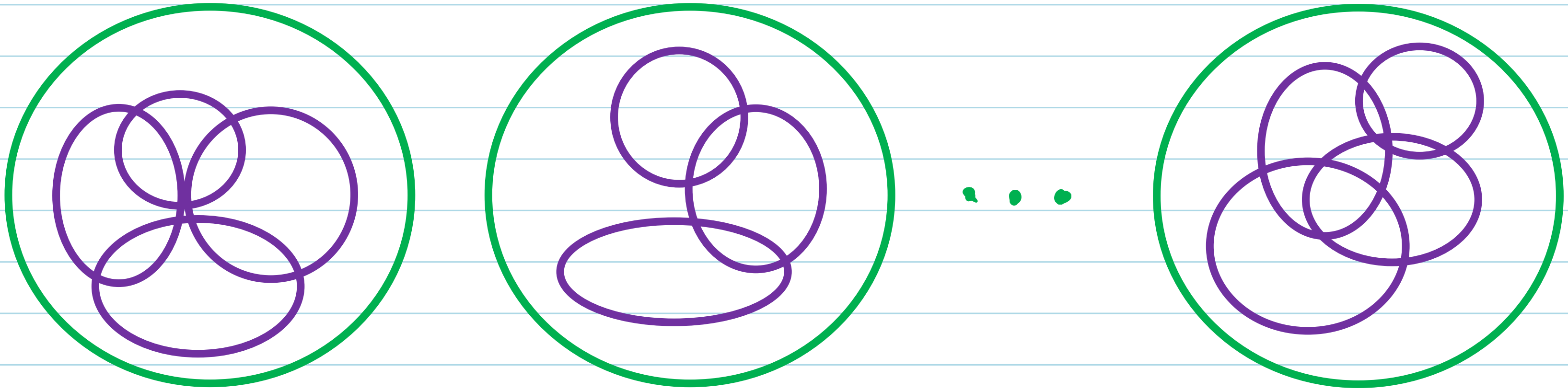
- Bound the # of antichains/container

Sketch of the Container Method

Container Theorem:

If the Boolean lattice is sufficienty "dense", then there exists containers $\mathcal{C}$ s.t.:



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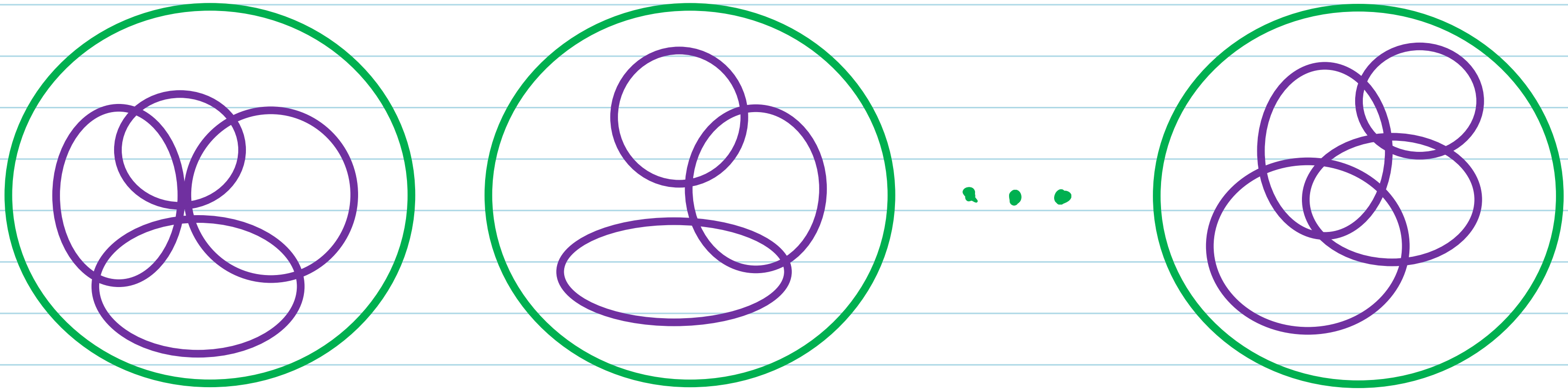

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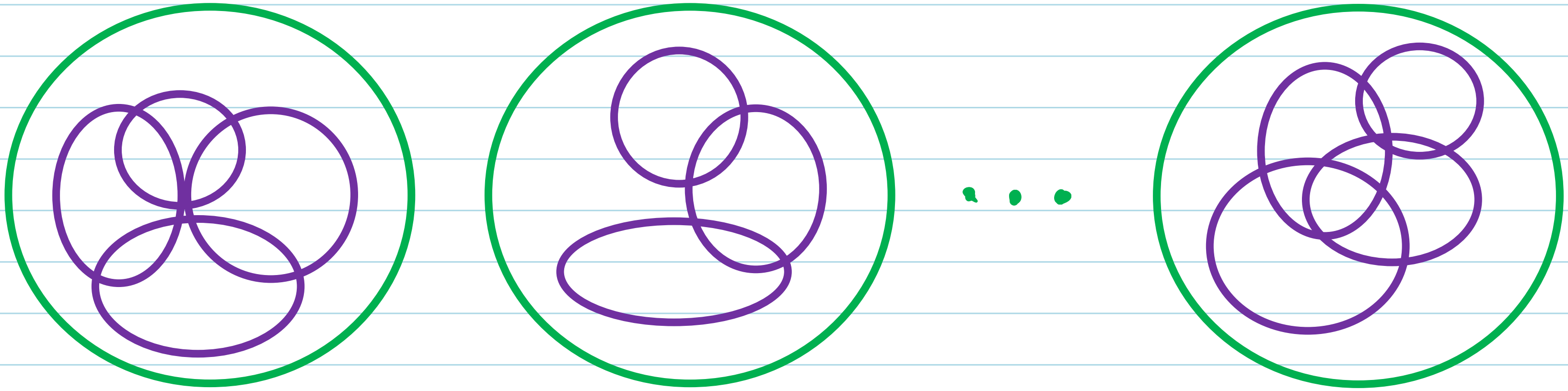
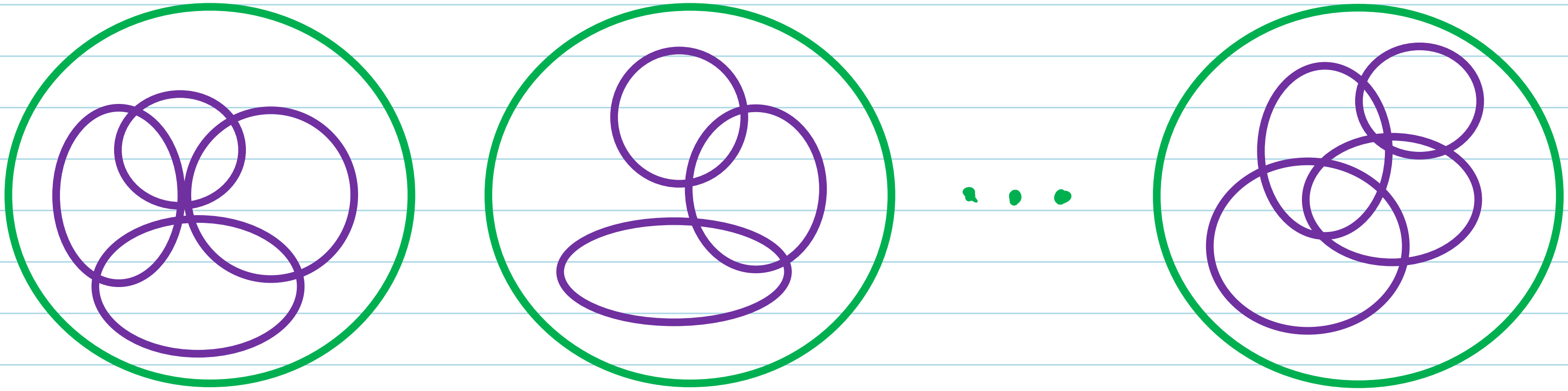


Sketch of the Container Method

Container Theorem:

If the Boolean lattice is sufficienty "dense", then there exists containers $\mathcal{C}$ s.t.:

- every antichain is a subset of a container,
- # containers is "small", and
- size of each container is "small".



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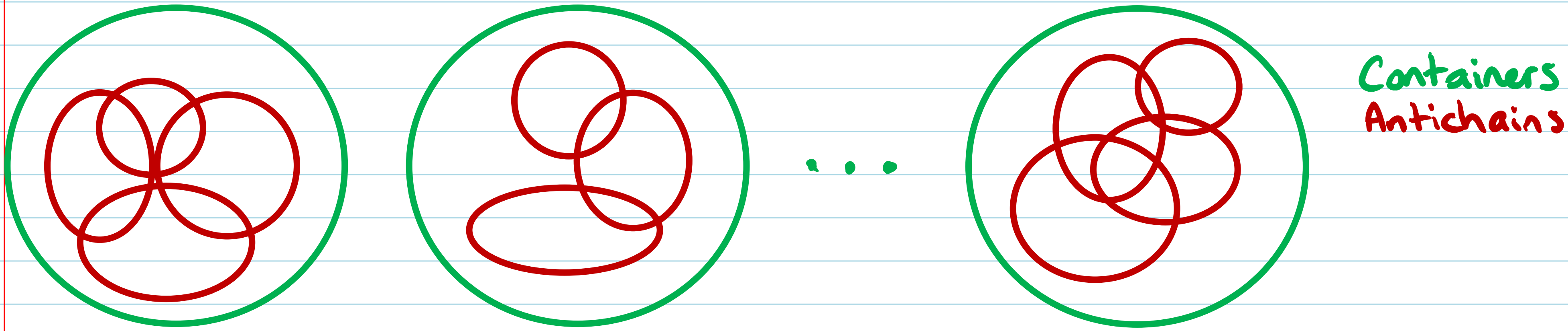

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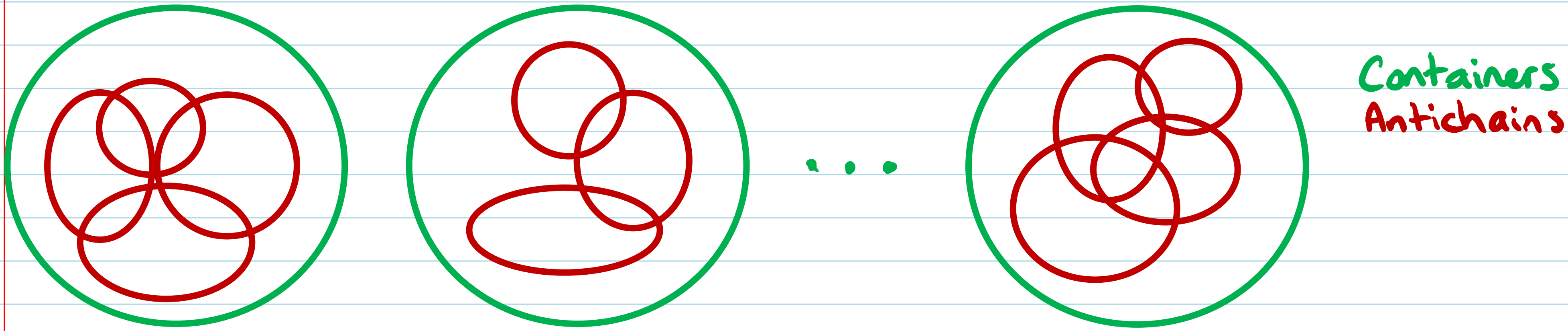

If we can prove the container theorem:

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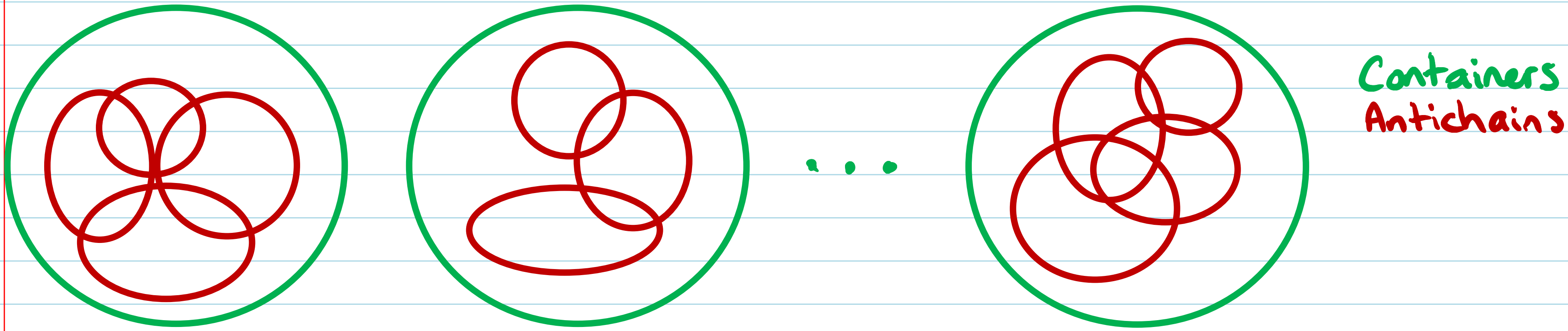

If we can prove the container theorem:

$$\# \text{antichains} \leq \sum_{C \in \mathcal{C}} 2^{|C|}$$

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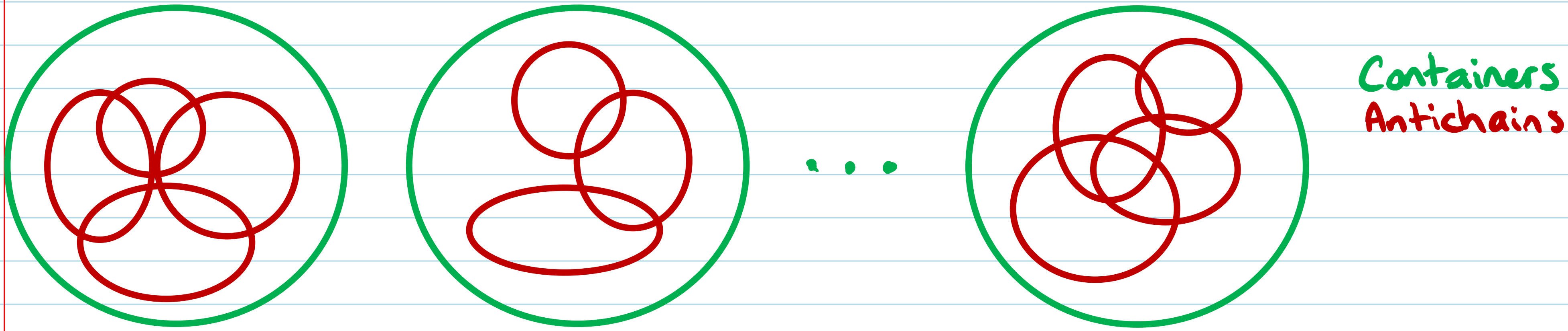

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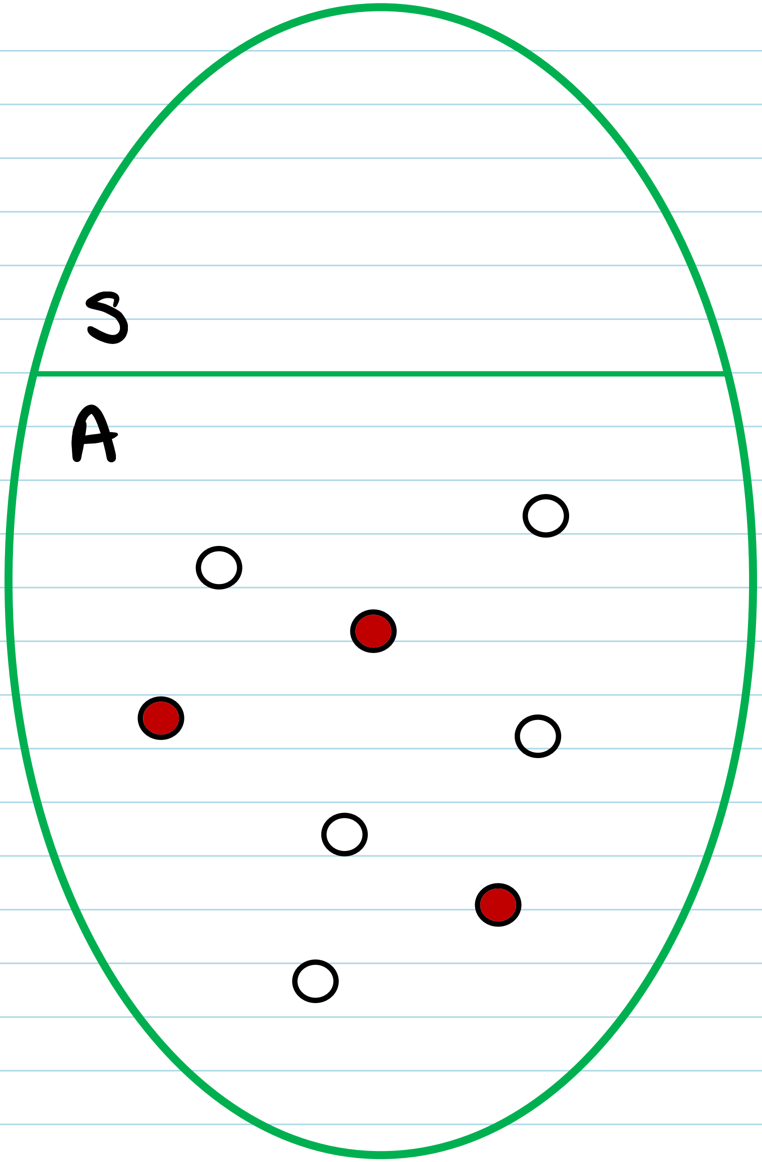
$$\leq |\mathcal{C}| \cdot 2^{\max |C|}$$

Proof of the Container Method

Scythe Algorithm

Input: $\mathcal{P}(n)$, $q \in \mathbb{Z}^+$, antichain I, $|I| \geq q$.

- $S = \emptyset$, $A = \mathcal{P}(n)$



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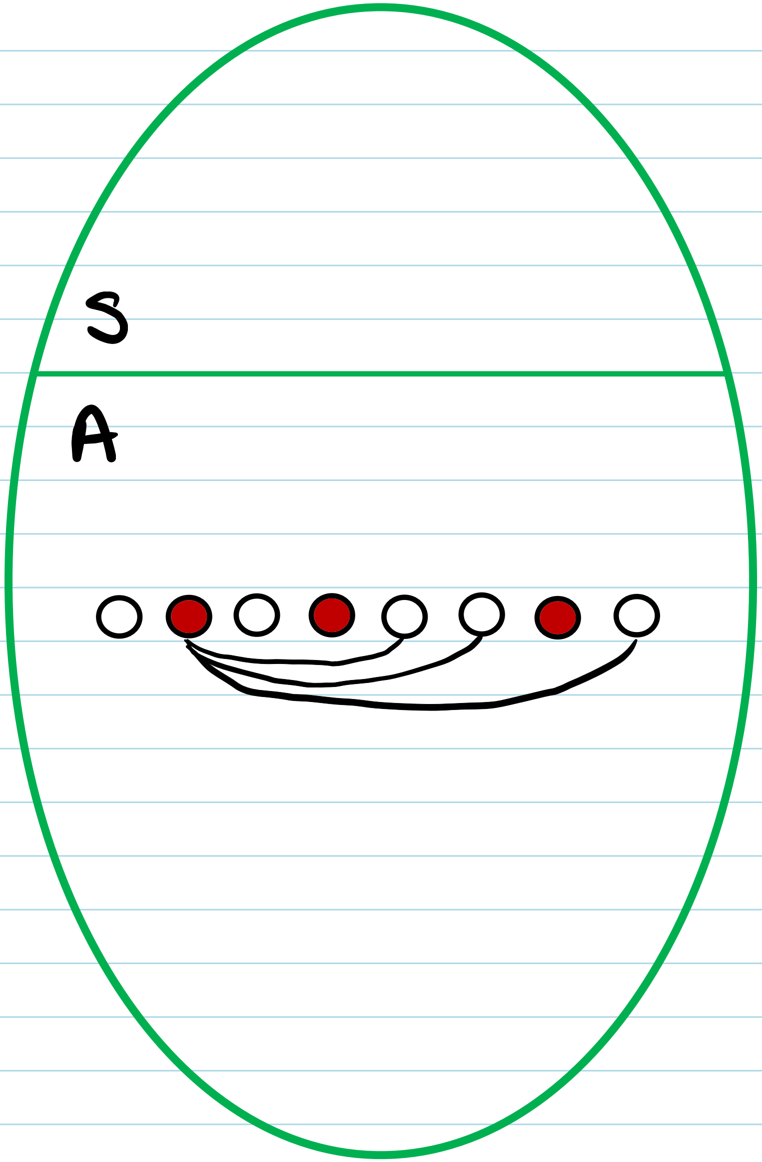

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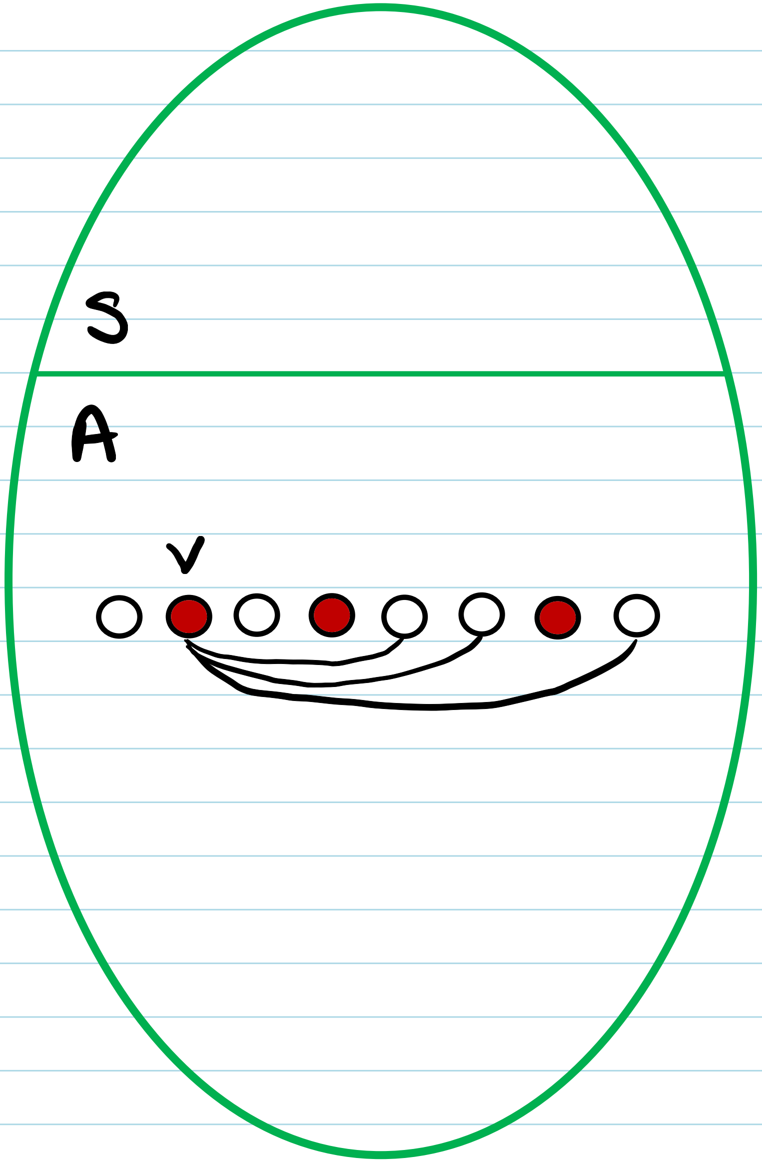

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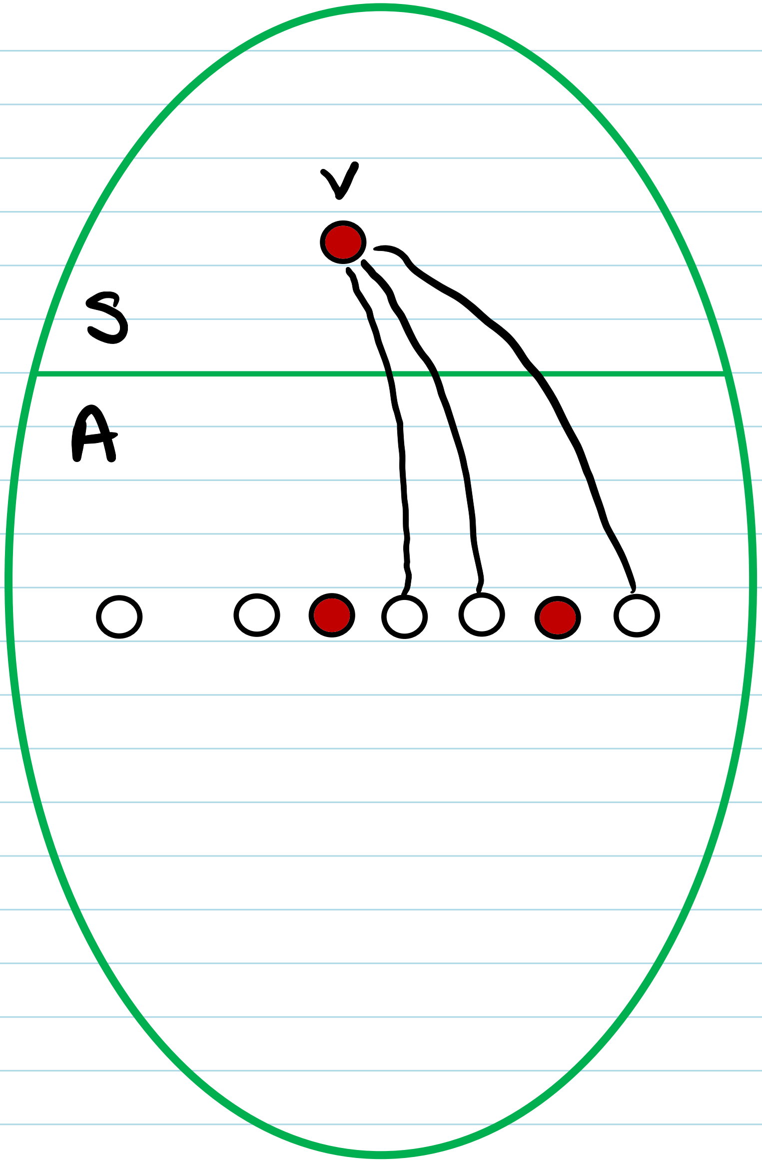

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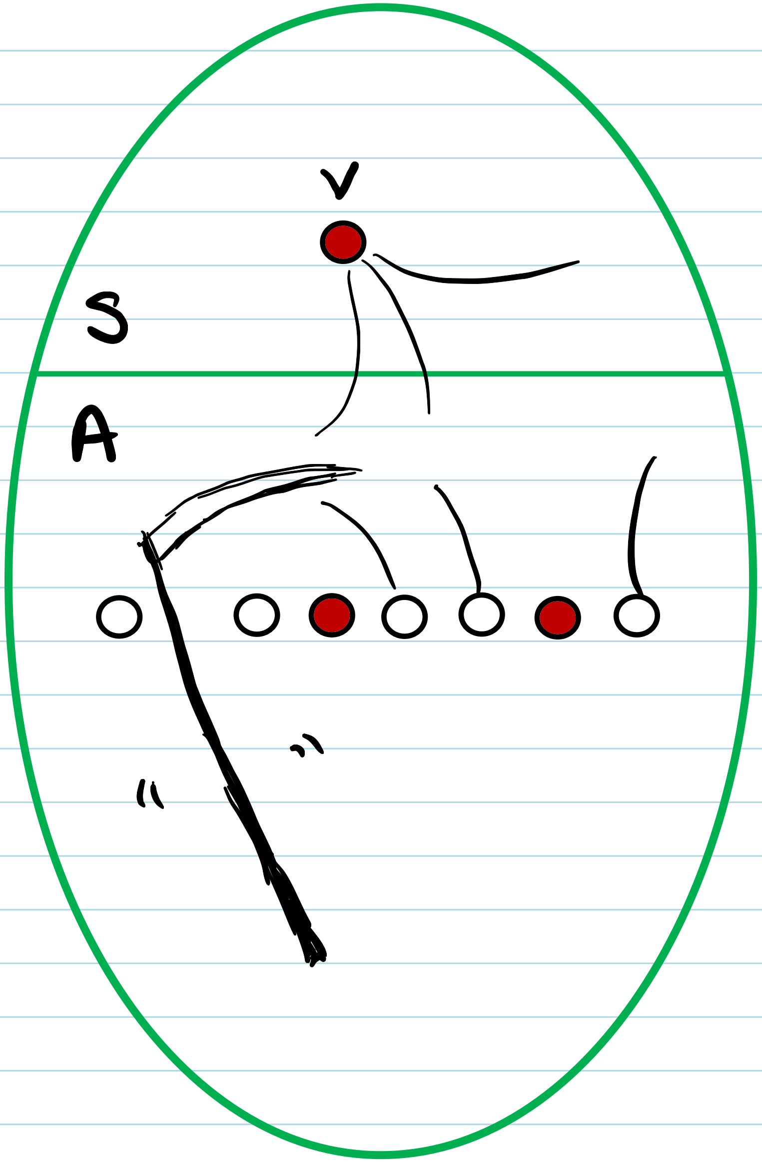

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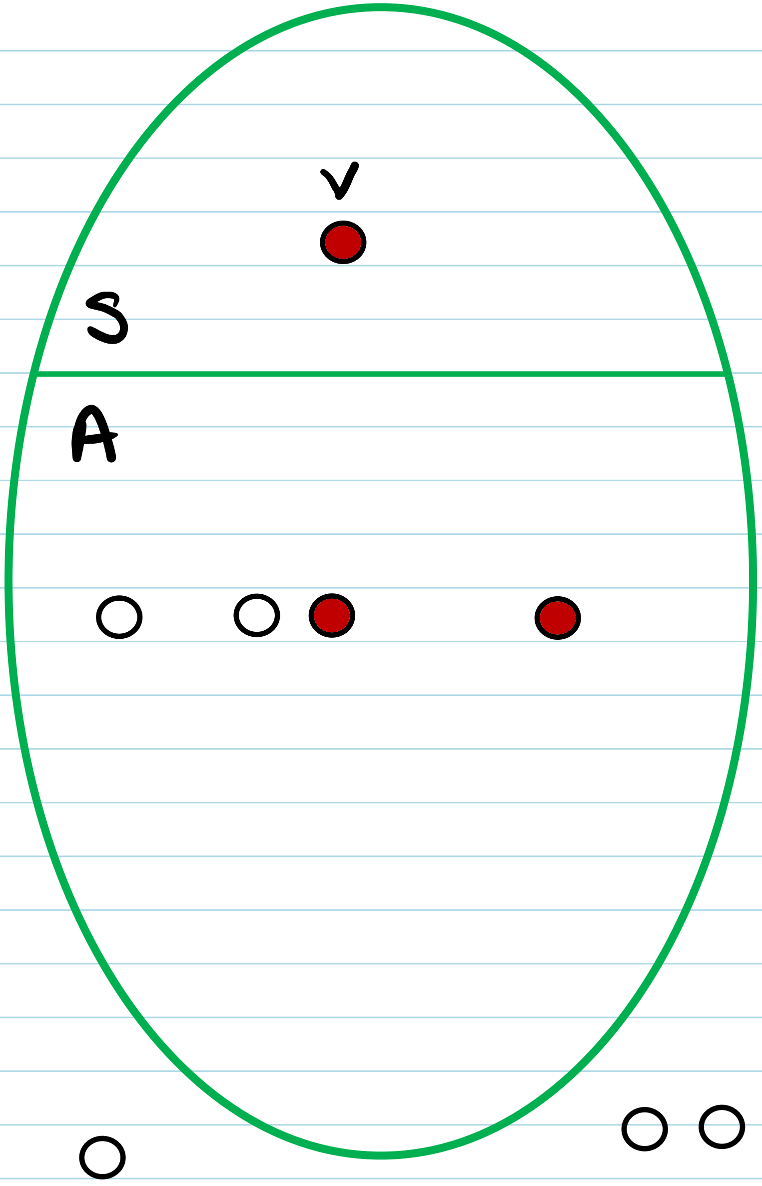

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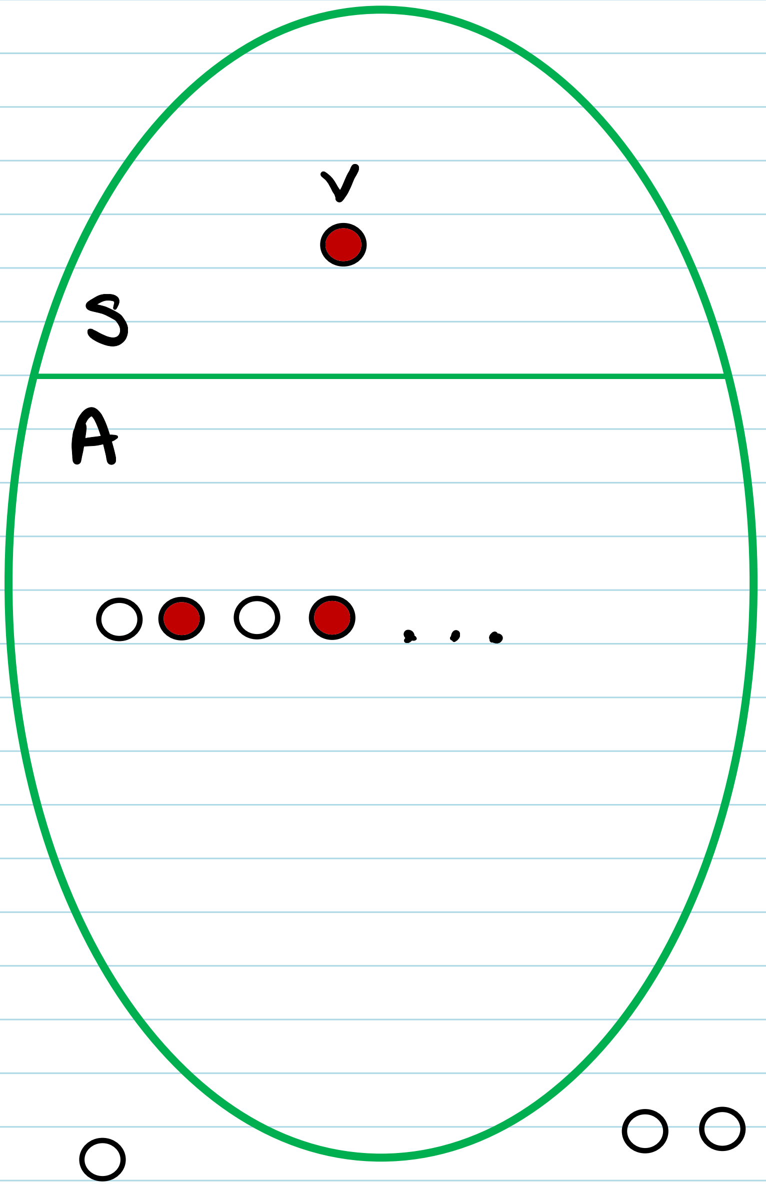

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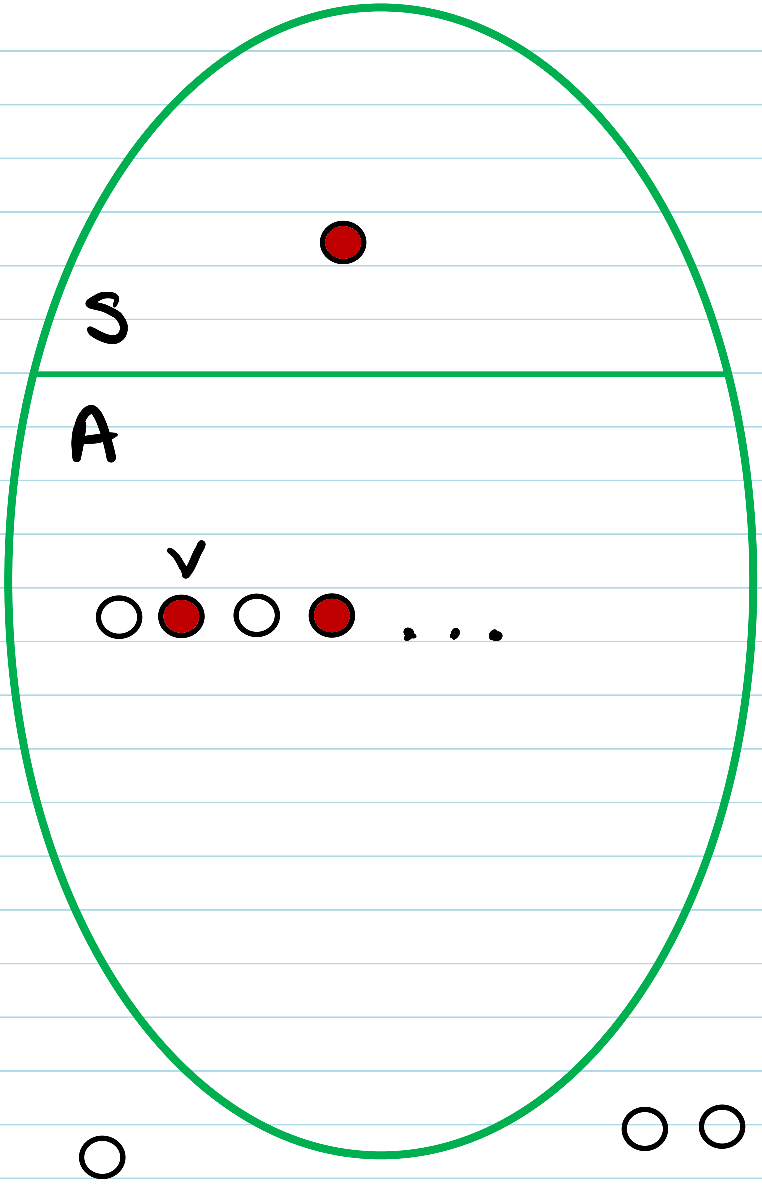

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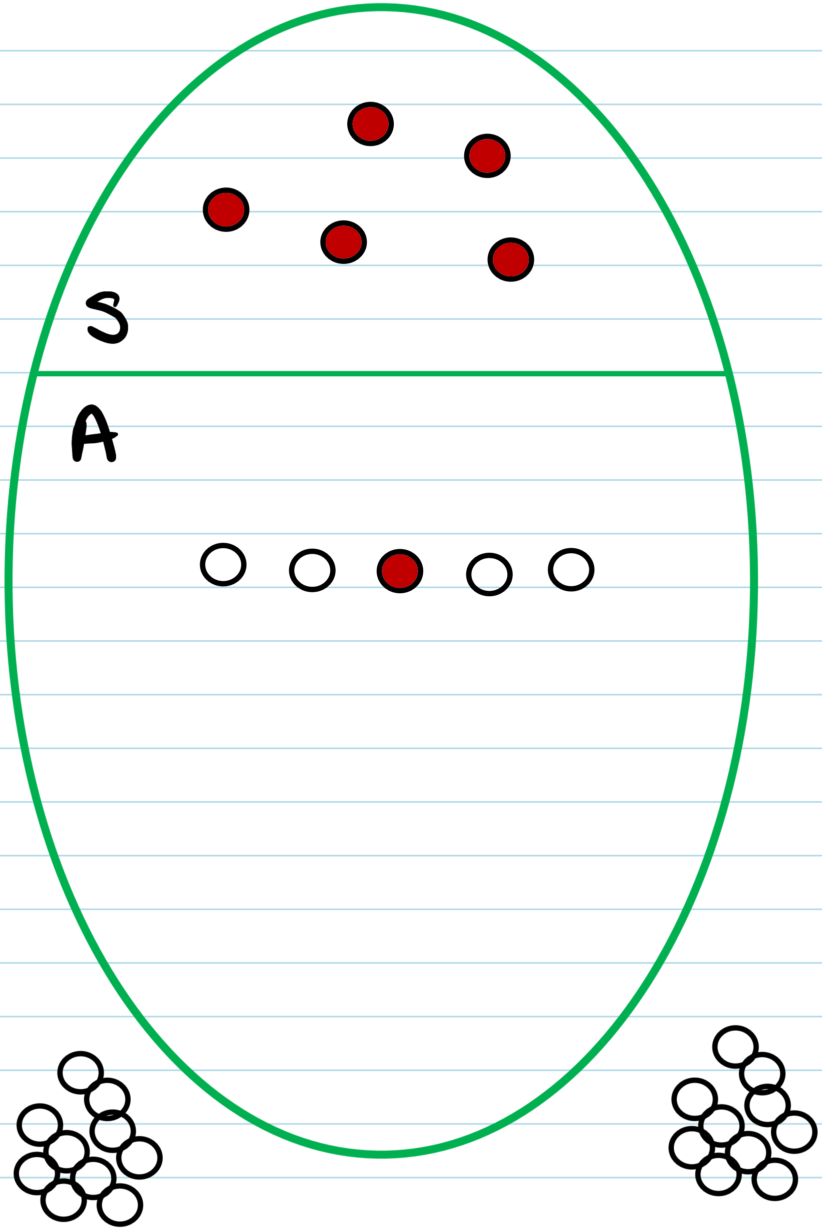
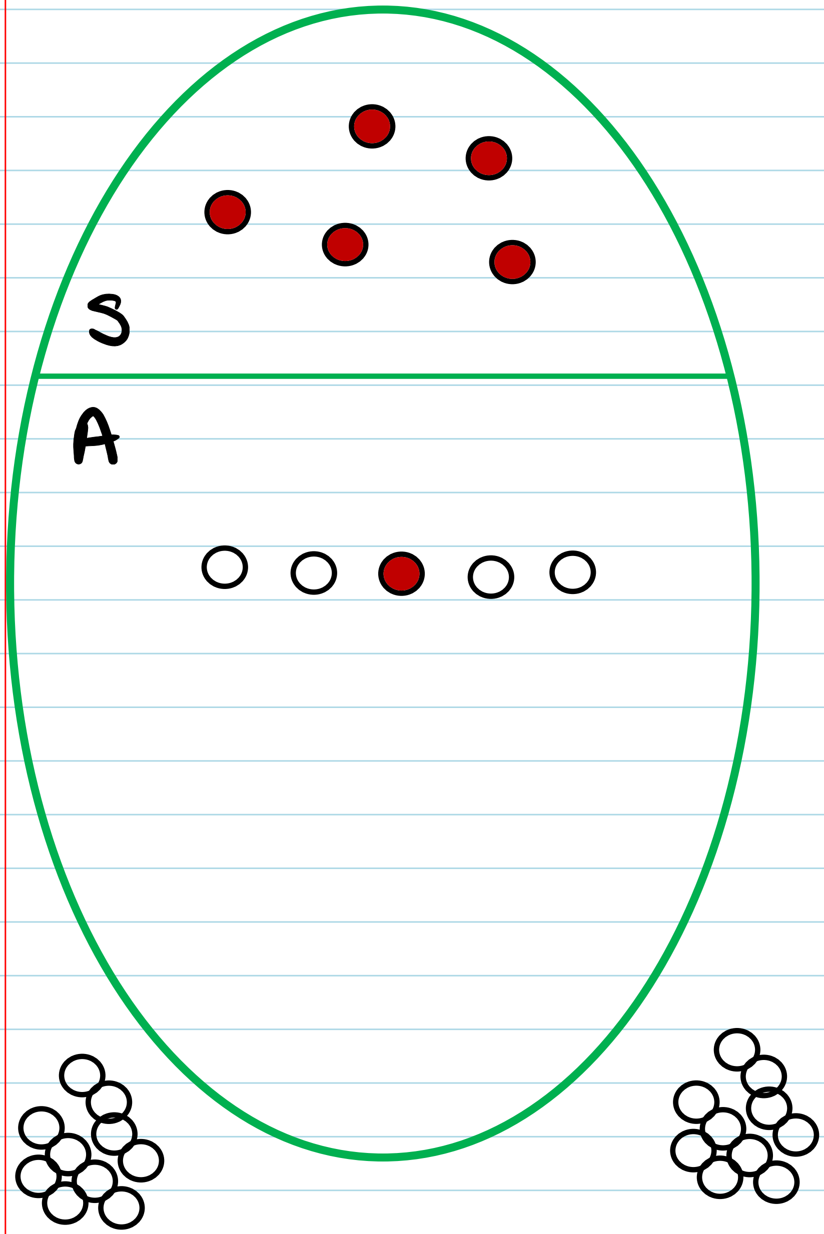
Output: (S, A)



Scythe Algorithm Observations



① $S \subseteq I$

fingerprint

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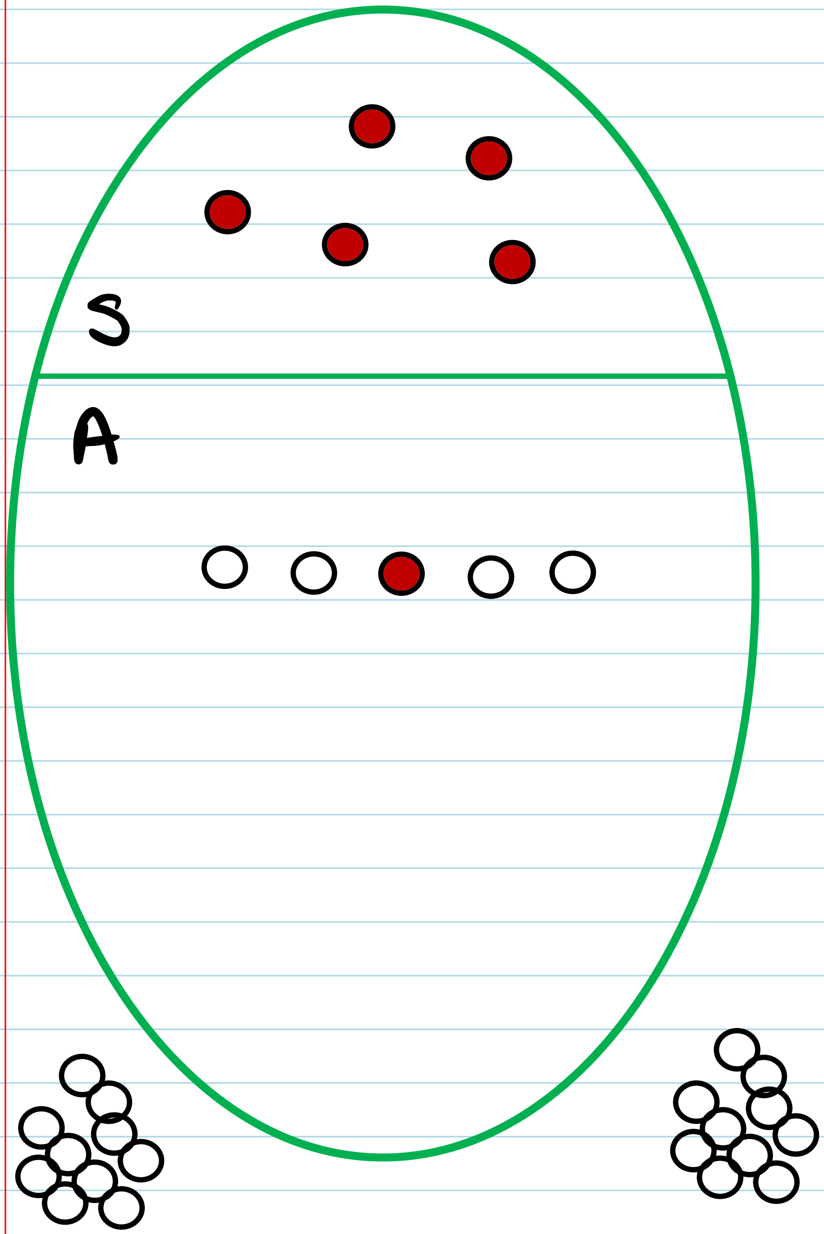

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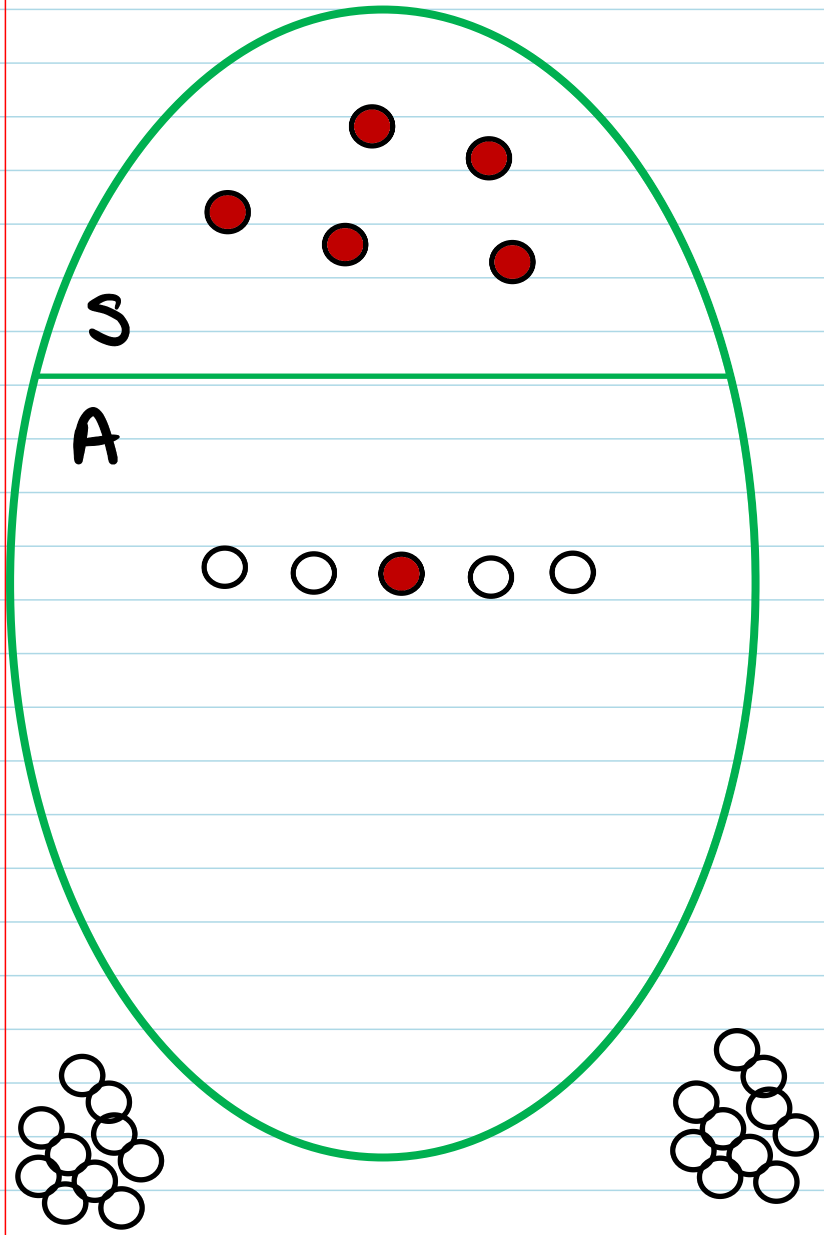

① $S \subseteq I$

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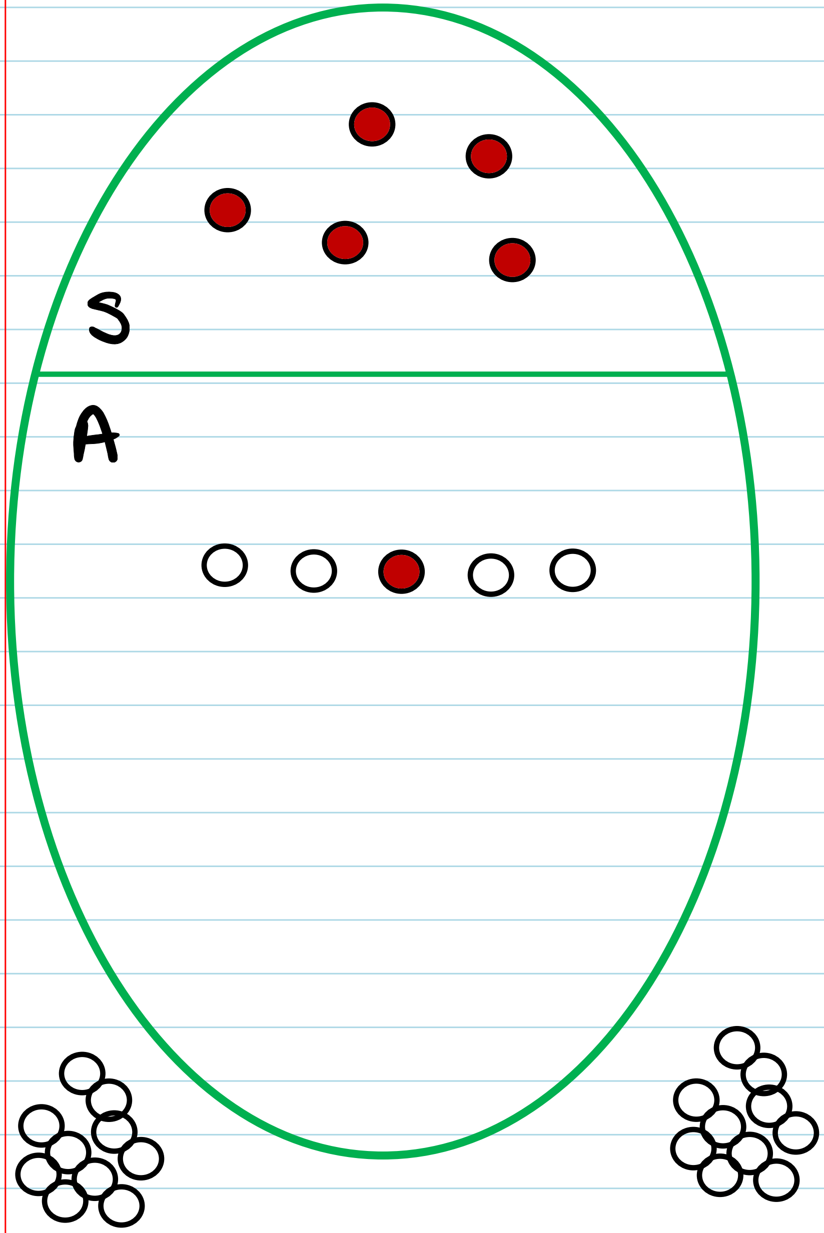
③ If $I \to (S, A)$, then

$S \to (S, A)$

fingerprints ⟷ containers

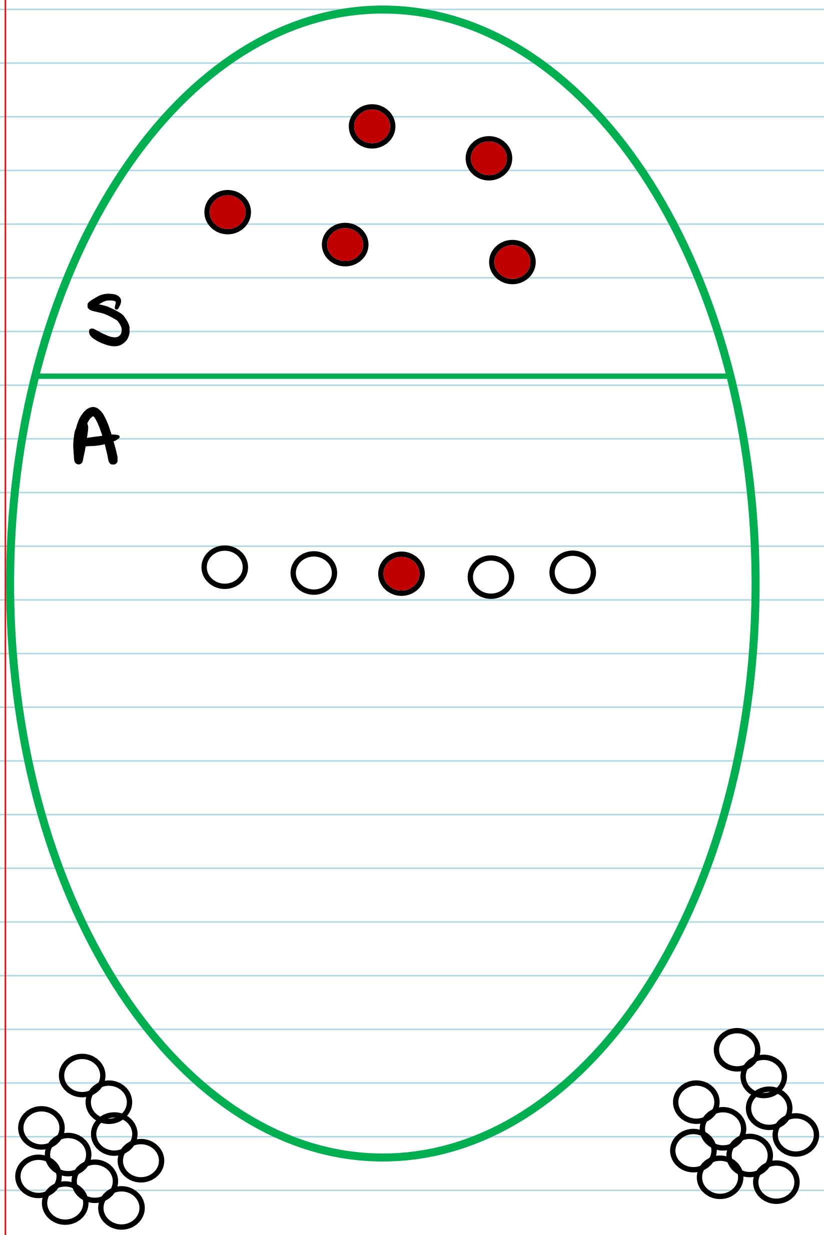
Scythe Algorithm Observations



④ $|S| = q$

$\Rightarrow$ # fingerprints $\leq \binom{2^n}{q}$

Scythe Algorithm Observations



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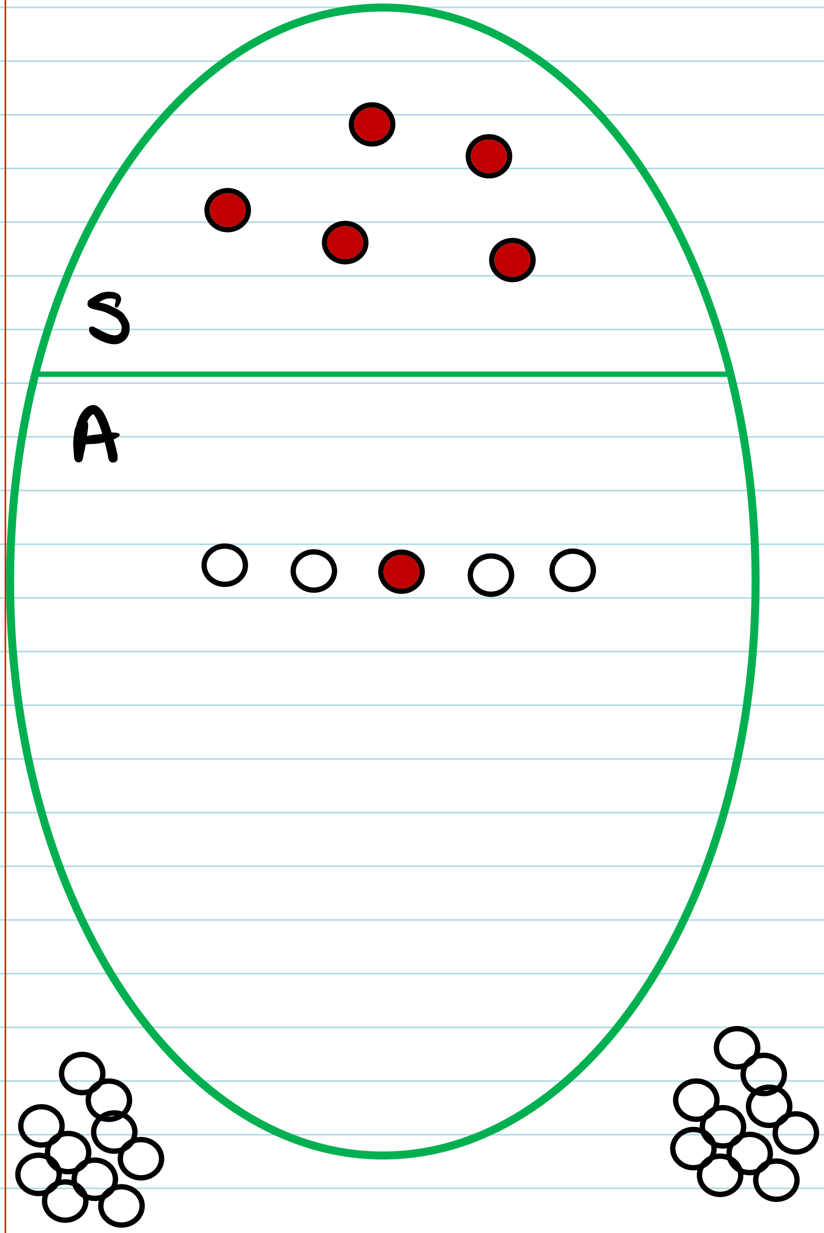
based on "density" assumption

Scythe Algorithm Observations



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⑤ $|A| \leq R$

based on "density" assumption

⑥ Antichains with size $< q$ not counted

$\leq \sum_{i=1}^{q-1} \binom{2^n}{i}$ small antichains

Putting it all together

containers

- fingerprints S ⟷ containers SUA

Putting it all together

containers

- fingerprints $S \longleftrightarrow$ containers $S \cup A$
- # containers = # fingerprints $\leq \binom{2^n}{q}$

Putting it all together

containers

- fingerprints $S \longleftrightarrow$ containers $S \cup A$
- $\#$ containers $=$ $\#$ fingerprints $\leq \binom{2^n}{q}$
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  - ↳ put each in its own container

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$$\# \text{containers} \leq \sum_{i=1}^{q-1} \binom{2^n}{i} + \binom{2^n}{q}$$

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$$\# \text{containers} \leq \sum_{i=1}^{q-1} \binom{2^n}{i} + \binom{2^n}{q}$$

$$= \sum_{i=1}^{q} \binom{2^n}{i}$$

Putting it all together

Size of a container

- containers are SUA

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$$\text{container size} \leq |A| + |S|$$

Putting it all together

Size of a container

- containers are SUA
- $|S| = q$
- $|A| \leq R$

↳ by the "density" assumption

$$\text{container size} \leq |A| + |S|$$
$$\leq R + q$$

Putting it all together

antichains

- # containers $\le \sum_{i=1}^{q} \binom{2^n}{i}$
- container size $\le R + q$

Putting it all together

\# antichains

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\# antichains $\leq |\mathcal{C}| \cdot 2^{\max |C|}$

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$$\leq \sum_{i=1}^{q} \binom{2^n}{i} \cdot 2^{R+q}$$

Putting it all together

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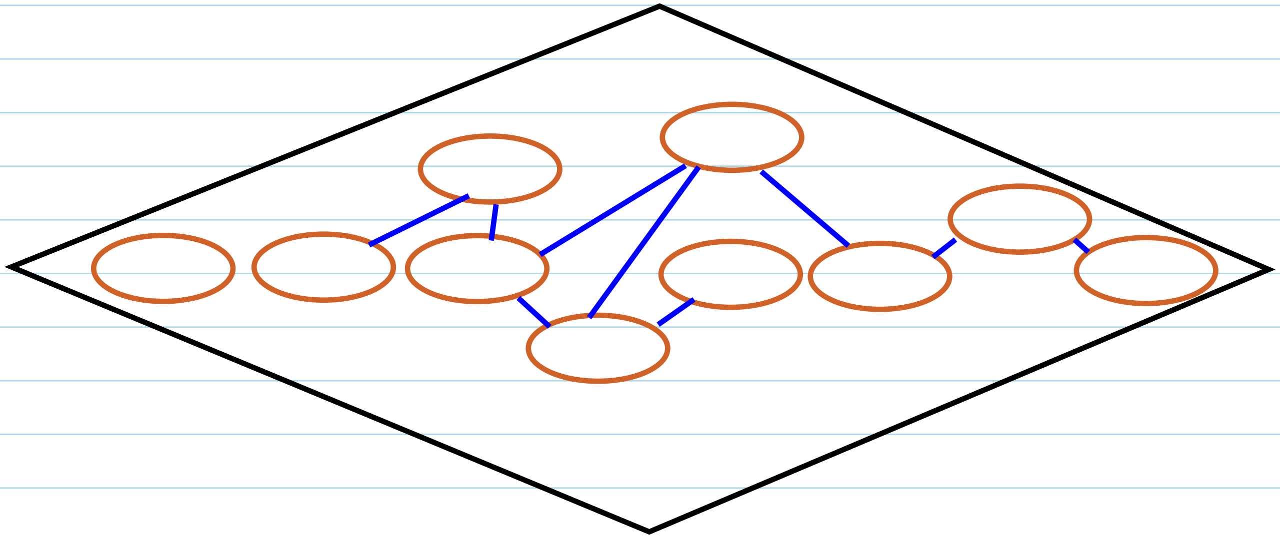
$$\# \text{antichains} \leq |\mathcal{C}| \cdot 2^{\max |C|}$$
$$\leq \sum_{i=1}^{q} \binom{2^n}{i} \cdot 2^{R+q}$$

Question: What are R and q?
Can we choose them?

Supersaturation Lemma

Supersat. for $\mathcal{P}(n)$

If $U \subseteq \mathcal{P}(n)$ has size $(1+\varepsilon)\binom{n}{n/2}$,
then # comparable pairs $\geq \beta\binom{|U|}{2}$.



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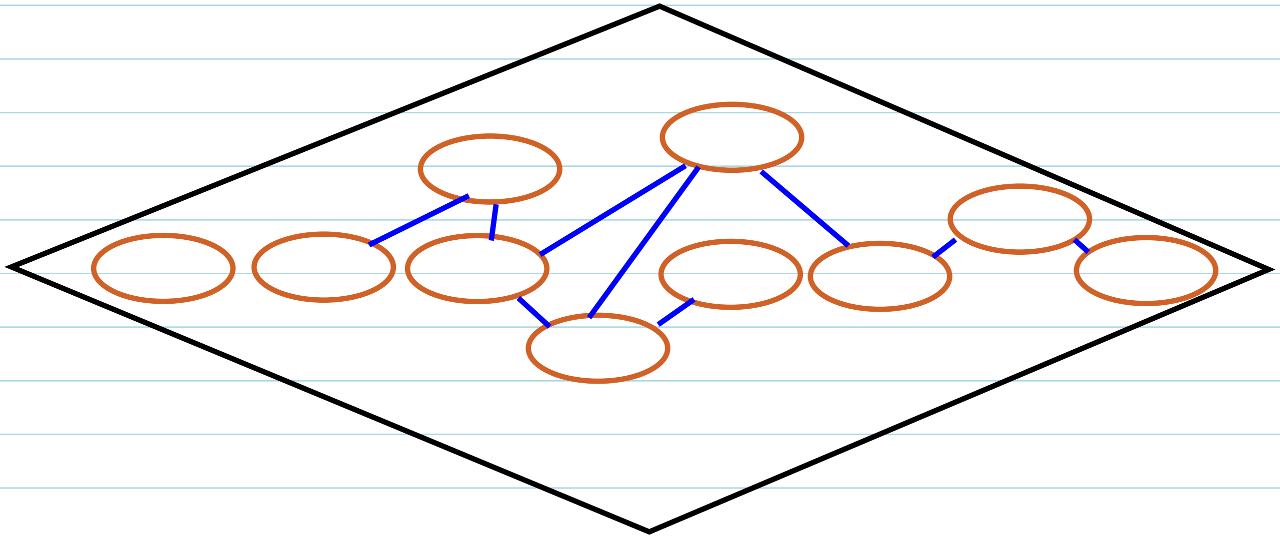
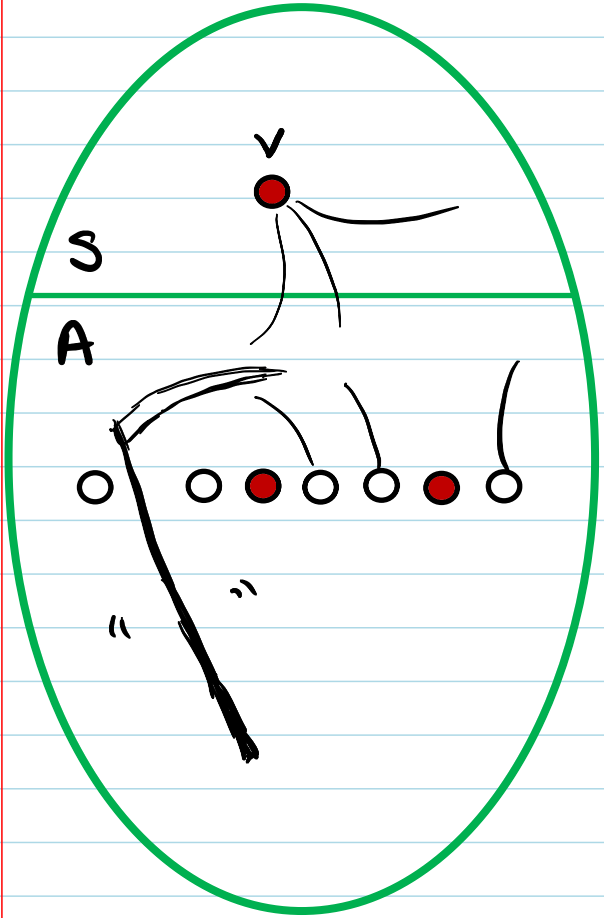


Goal: Satisfy the "density" condition of the Container Theorem.

Why Supersaturation?



Supersat. Lemma

$|U| = (1+\varepsilon)\binom{n}{n/2} \geq R \Rightarrow$

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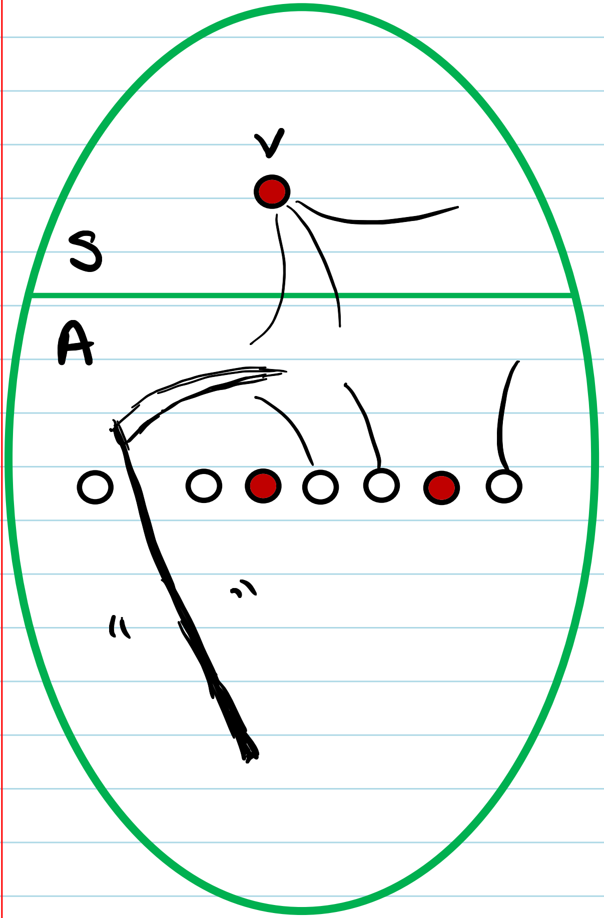

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$\Downarrow$

A β-fraction of A is removed in each loop.

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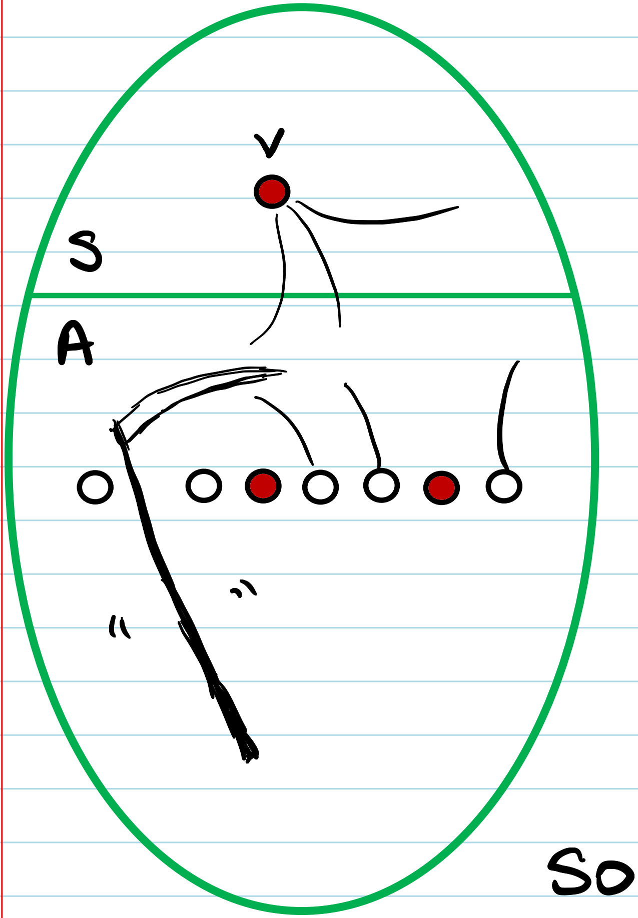

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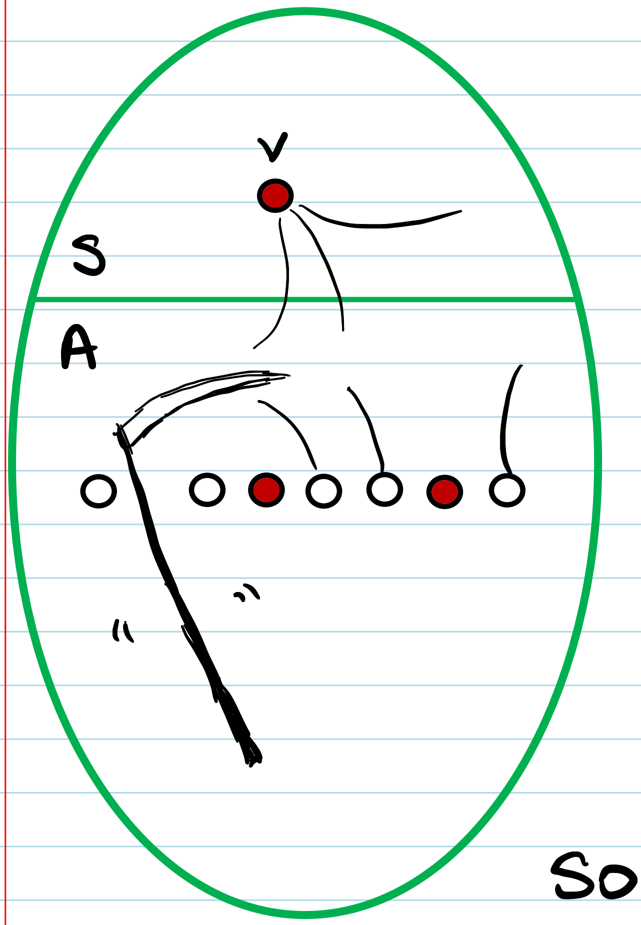
$$\#\text{ comparable pairs} \geq \beta\binom{|U|}{2}$$

$$\Downarrow$$

A β-fraction of A is removed in each loop.

So... $|A| \leq (1-\beta)^q \cdot 2^n \leq e^{-\beta q} \cdot 2^n$

Why Supersaturation?



Supersat. Lemma

$|U| = (1+\varepsilon)\binom{n}{n/2} \geq R \Rightarrow$

$\#$ comparable pairs $\geq \beta\binom{|U|}{2}$

$\Downarrow$

A β-fraction of A is removed in each loop.

So... $|A| \leq (1-\beta)^{q} \cdot 2^{n} \leq e^{-\beta q} \cdot 2^{n}$

Goal: Choose R, q, β s.t. $e^{-\beta q} \cdot 2^{n} \leq R$

("density" condition)

Choosing R, q, and β

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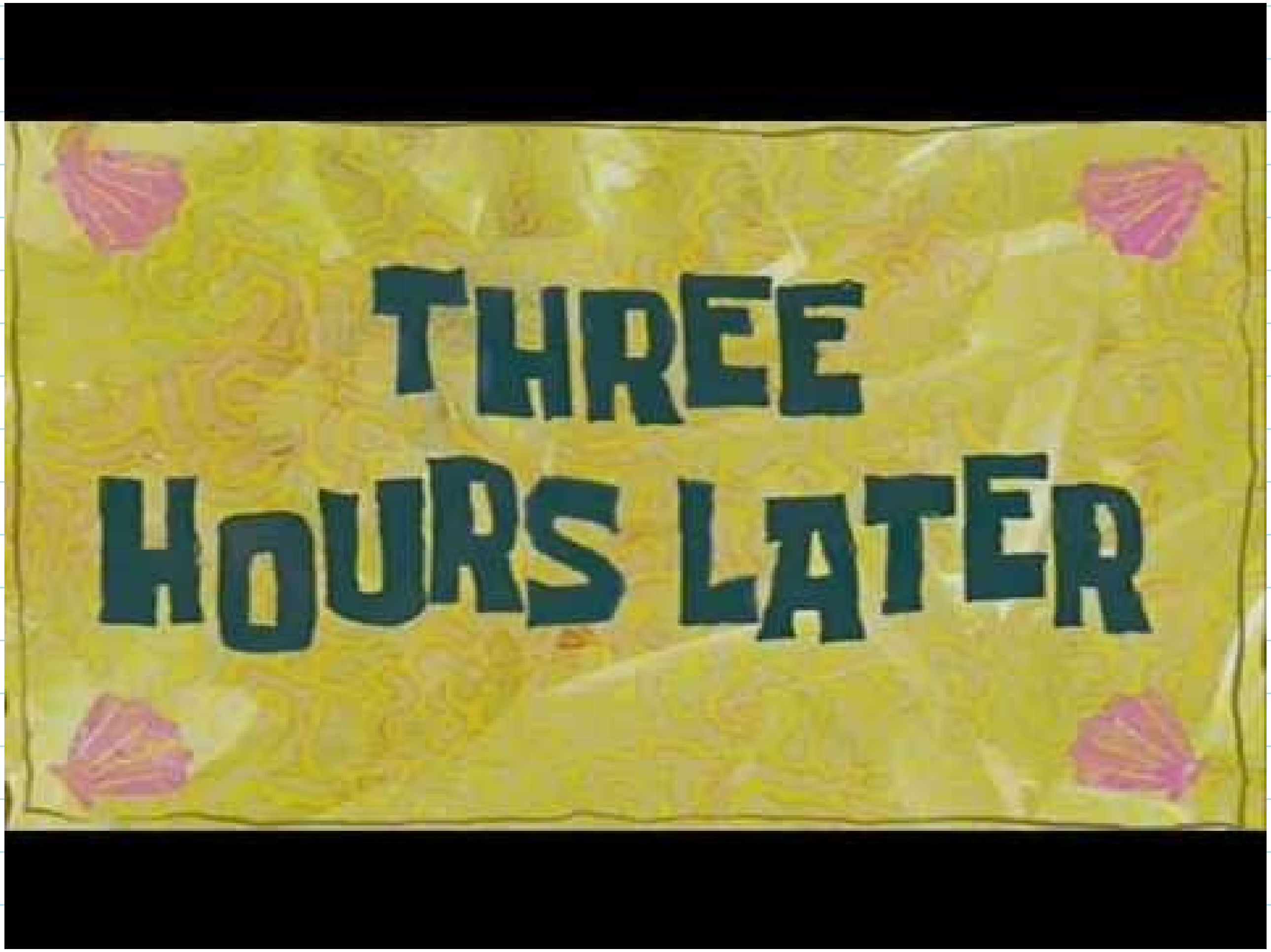
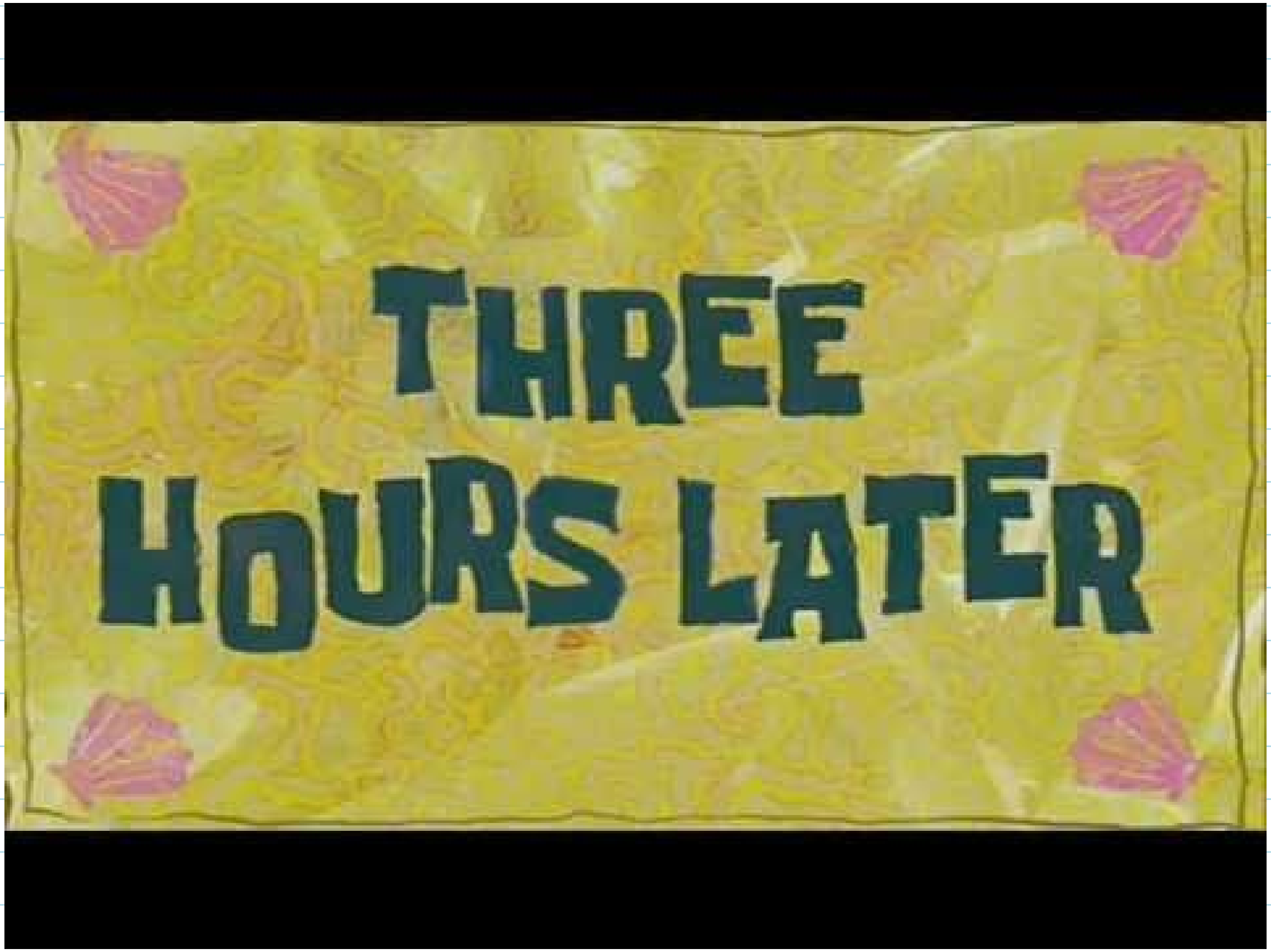


Choosing R, q, and β



$R = \left(1 + \frac{1}{\sqrt{n}}\right)\binom{n}{n/2}$ $q = \frac{\log n}{n} 2^n$ $\beta = \frac{n}{2^n}$

Antichain Upper Bound

Ingredients

- $R = \left(1 + \frac{1}{\sqrt{n}}\right)\binom{n}{n/2}$
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$$\le 2^{\binom{n}{n/2}\left(1 + \frac{1+\log n + \log^2 n}{\sqrt{n}}\right)}$$

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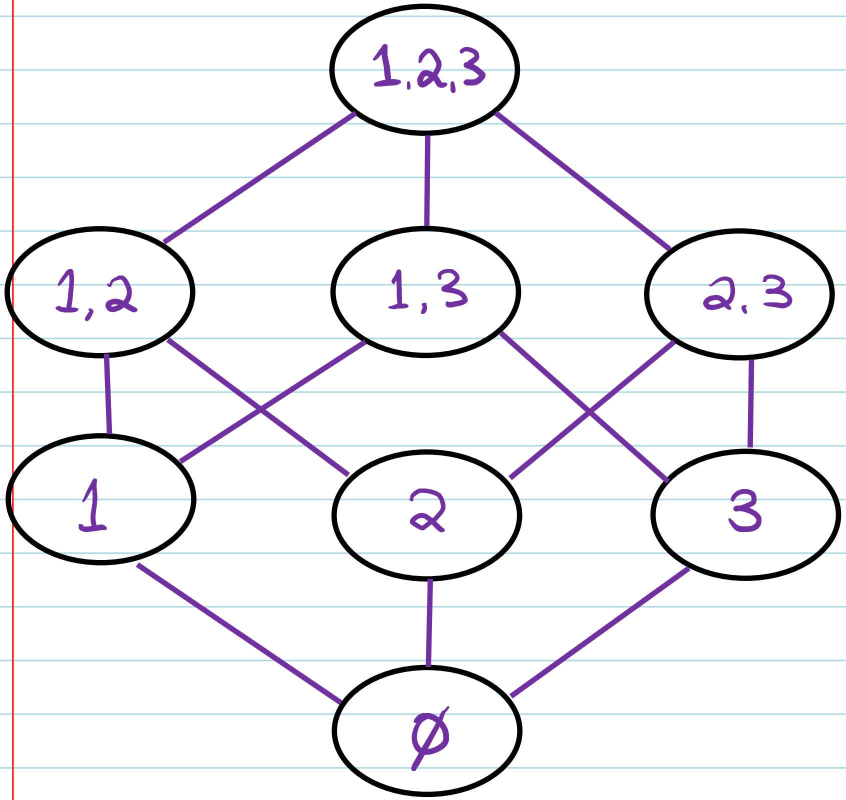
$$\le 2^{\binom{n}{n/2}\left(1+\frac{1+\log n+\log^2 n}{\sqrt{n}}\right)}$$

$$= 2^{\binom{n}{n/2}(1+o(1))}$$

Number of Antichains

- Lower Bound: $2^{\binom{n}{n/2}}$
- Upper Bound: $2^{\binom{n}{n/2}(1+o(1))}$

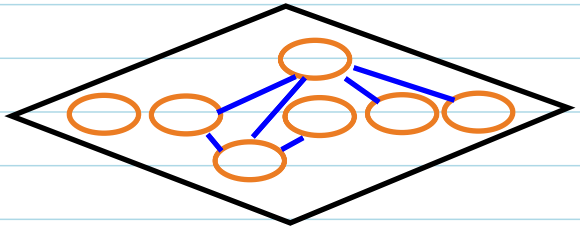


$$\#\text{antichains in } \mathcal{P}(n) = 2^{\binom{n}{n/2}(1+o(1))}$$

☐

Container Method Summary

① Prove a Supersaturation Lemma.



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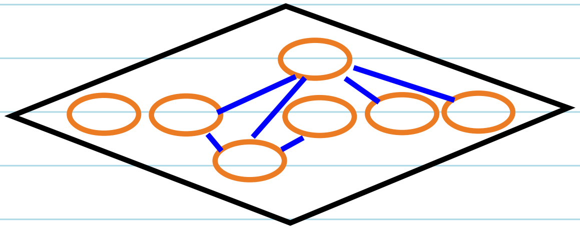

② Satisfy the "density" condition

$R \qquad q \qquad \beta$

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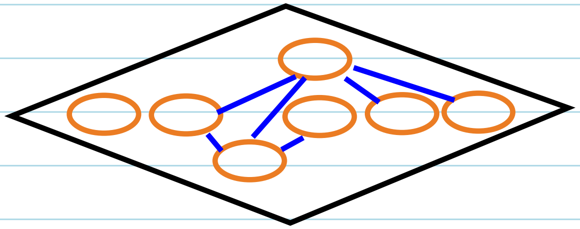

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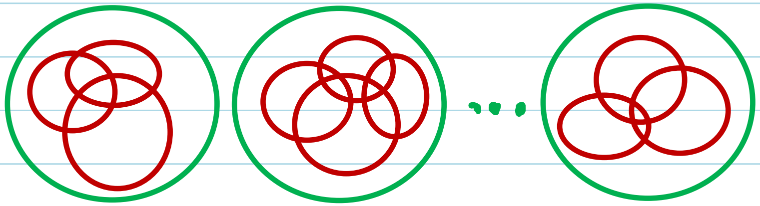

$$\# \le \sum_{i=1}^{q} \binom{2^n}{q} \cdot 2^{R+q}$$

A Surprising Application

- matroids $M = (E, \mathcal{I})$

  ground set ↑ ↑ independent sets

e.g.

$$\begin{array}{c} E \\ \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix} \end{array}$$

$I \subseteq E$ linearly indep.

$\Rightarrow I \in \mathcal{I}$.

A Surprising Application

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  $$\begin{array}{c} E \\ \left[\begin{matrix} \quad & \quad & \quad \\ & & \\ & & \end{matrix} \right] \end{array}$$
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- a matroid extension is obtained by "adding" a new element.

A Surprising Application

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Lemma: Most extensions of clique matroids can be rep'd by *intersecting* antichains.

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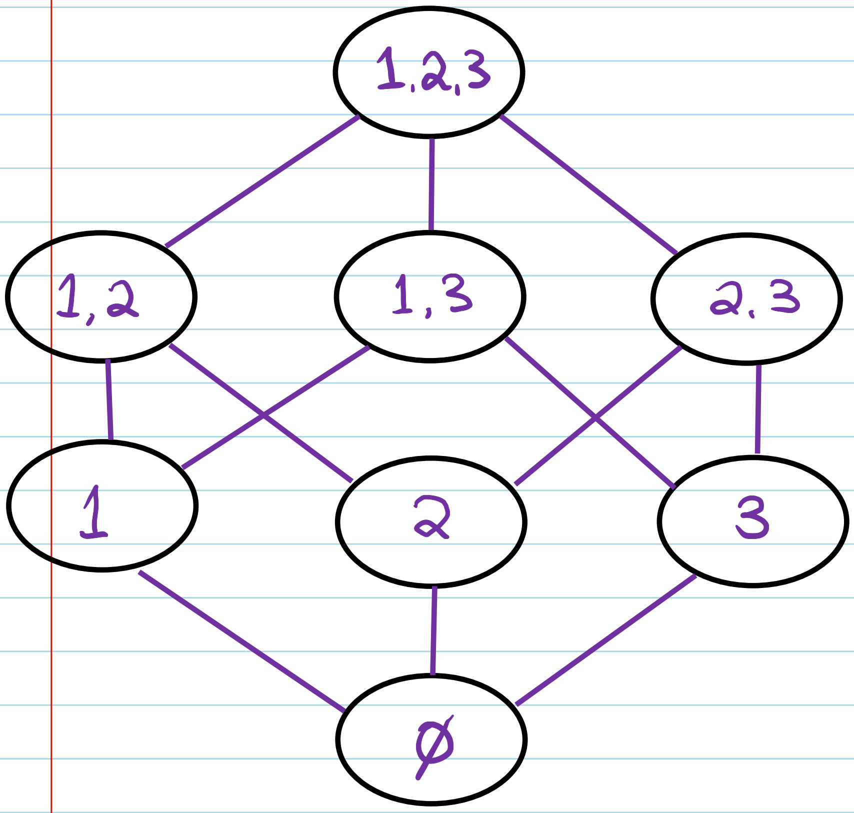
Lemma: Most extensions of clique matroids can be rep'd by *intersecting* antichains.

Theorem: $\#$ extensions $= 2^{\frac{1}{2}\binom{n}{n/2}(1+o(1))}$

Connection to Graphs



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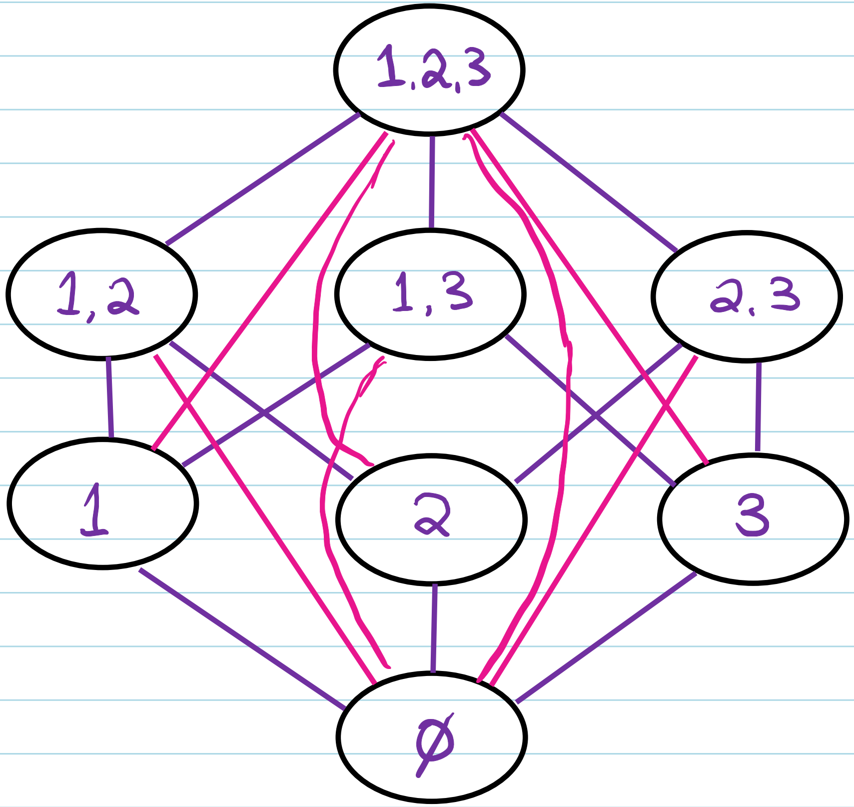
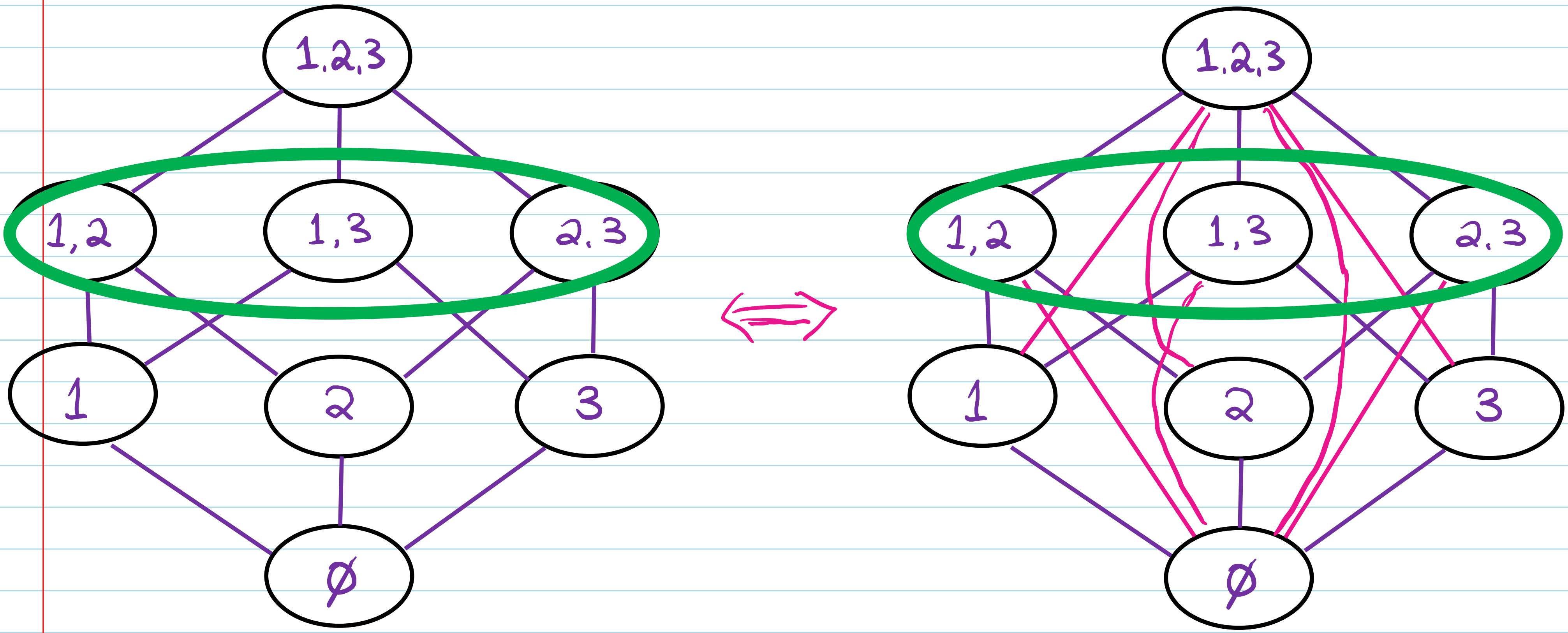


antichain:
no two elements
are comparable.

independent set:
no two vertices
are adjacent.

Connection to Graphs



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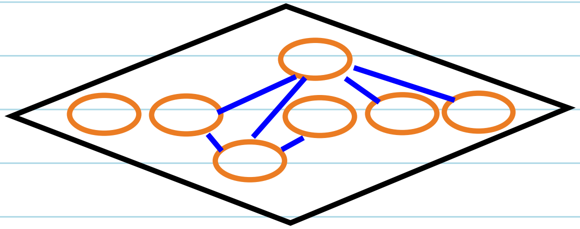

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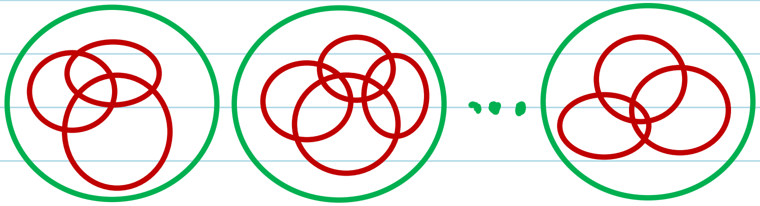

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