PMATH 340 Number Theory, Exercises for Chapter 8 (Diophantine Equations)

1: For each of the following integers n, determine whether n is a sum of two squares, and if so then find the number of pairs $(x, y) \in \mathbf{Z}^2$ for which $n = x^2 + y^2$.

(a) $n = 1081$

(b) $n = 3{,}185{,}000$

(c) $n = \binom{100}{11} = \frac{100!}{11!\,89!}$

2: Let $n = 99450$.

(a) Write n as a product of irreducible elements in $\mathbf{Z}[i]$.

(b) List all of the pairs $(x, y) \in \mathbf{Z}^2$ with $0 \leq x \leq y$ such that $n = x^2 + y^2$.

3: (a) Solve Pell's equation $x^2 - 22y^2 = 1$.

(b) Solve Pell's equation $x^2 - 13y^2 = 1$.

4: (a) Let $d \in \mathbf{Z}^+$ be a non-square and let $0 \neq n \in \mathbf{Z}$. Show that the Diophantine equation $x^2 - dy^2 = n$ either has no solution or infinitely many solutions.

(b) For which $n \in \mathbf{Z}$ with $-3 \leq n \leq 10$ do there exist $x, y \in \mathbf{Z}$ with $x^2 - 31\,y^2 = n$?

5: (a) Find the first 2 smallest (and possibly only) positive solutions to the Diophantine equation $x^2 - 2y^4 = -1$.

(b) Find the first 4 smallest positive solutions to the Diophantine equation $x(x + 1) = 2y^2$.