PMATH 340 Number Theory, Exercises for Chapter 7 (Continued Fractions)

1: (a) Express the (finite) continued fraction $[2,1,3,1,2]$ as a rational number, in reduced form.
(b) Express that rational number $\frac{64}{47}$ as a (finite) continued fraction.

2: (a) Express the (periodic) continued fraction $[1,\overline{1,3}\cdots]$ as a quadratic irrational.
(b) Express the quadratic irrational $\frac{3+\sqrt{7}}{2}$ as a (periodic) continued fraction.

3: (a) Express $\sqrt{7}$ as a continued fraction and find the the k^{th} convergents $c_k = \frac{p_k}{q_k}$ for $0 \leq k \leq 7$. Let $u_k = p_k + q_k\sqrt{7} \in \mathbf{Z}[\sqrt{7}]$ for $0 \leq k \leq 7$ and calculate $u_3 u_k \in \mathbf{Z}[\sqrt{7}]$ for $0 \leq k \leq 3$. What do you notice?
(b) Express $\sqrt{2}$ as a continued fraction then show that the k^{th} convergent is given by $c_k = \frac{p_k}{q_k}$ with

$$p_k = \tfrac{1}{2}\Big((1+\sqrt{2})^{k+1} + (1-\sqrt{2})^{k+1}\Big) \quad\text{and}\quad q_k = \tfrac{1}{2\sqrt{2}}\Big((1+\sqrt{2})^{k+1} - (1-\sqrt{2})^{k+1}\Big).$$

4: (a) Express $\sqrt{57}$ as a continued fraction and find the smallest unit $u > 1$ in $\mathbf{Z}\big[\sqrt{57}\,\big]$.
(b) Determine whether 5 is irreducible in the ring $\mathbf{Z}[\sqrt{57}\,]$.

5: (a) Let $x = [a_0, a_1, a_2, \cdots]$ with $a_0 \in \mathbf{Z}$ and $a_i \in \mathbf{Z}^+$ for $i \geq 2$. Show that

$$-x = \begin{cases} \big[-a_0 - 1, a_2 + 1, a_3, a_4, a_5, \cdots\big] & \text{, if } a_1 = 1 \\ \big[-a_0 - 1\,,\, 1\,,\, a_1 - 1, a_2, a_3, a_4, \cdots\big] & \text{, if } a_1 > 1. \end{cases}$$

(b) Let $x = \sqrt{d}$, where $d \in \mathbf{Z}^+$ is a non-square, so we have $x = \big[a_0, \overline{a_1, a_2, \cdots, a_{\ell-1}, 2a_0}\,\big]$ where ℓ is the minimum period of the sequence $\{a_k\}$. Show that $\{a_k\}$ is symmetric in the sense that $a_k = a_{\ell-k}$ for $0 < k < \ell$.