PMATH 340 Number Theory, Solutions to the Exercises for Chapter 1

1: Let $a = 17537$, $b = 5434$ and $c = 1482$.

(a) Find $\gcd(a, b)$ and $\operatorname{lcm}(a, b)$.

Solution: We use the Euclidean Algorithm.

$$\begin{aligned} 17537 &= 3 \cdot 5434 + 1235 \\ 5434 &= 4 \cdot 1235 + 494 \\ 1235 &= 2 \cdot 494 + 247 \\ 494 &= 2 \cdot 247 + 0 \end{aligned}$$

This gives $\gcd(a, b) = 247$, and hence $\operatorname{lcm}(a, b) = \dfrac{ab}{\gcd(a, b)} = \dfrac{17537 \cdot 5434}{247} = 385814$.

(b) Solve the linear Diophantine equation $ax + by = c$.

Solution: Let $d = \gcd(a, b) = 247$. Note that $c = 6\,d$, so solutions do exist. Also note that $a = 71\,d$ and $b = 22\,d$. Back substitution gives the sequence $\{u_k\} = 1, -2, 9, -29$ and so $9a - 29b = d$. Multiply by 6 to get $54\,a - 174\,b = c$, so we have one solution one solution $(x_0, y_0) = (54, -174)$. The general solution is $(x, y) = (x_0, y_0) + k\left(-\frac{b}{d}, \frac{a}{d}\right)$ that is $(x, y) = (54, -174) + k\,(-22, 71)$, where $k \in \mathbf{Z}$.

2: (a) Find $\sigma(10!)$.

Solution: We have $10! = 2^{5+2+1} \cdot 3^{3+1} \cdot 5^2 \cdot 7^1 = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7^1$ and so

$$\sigma(10!) = (1 + 2 + 4 + 8 + \cdots + 256)(1 + 3 + 9 + 27 + 81)(1 + 5 + 25)(1 + 7) = 511 \cdot 121 \cdot 31 \cdot 8 = 15334088.$$

(b) List all of the positive integers n such that $\sigma(n) = 42$.

Solution: We begin by listing $\sigma(p^k)$ for all prime powers p^k for which $\sigma(p^k)\big|42$:

$$\sigma(2) = 3\ ,\ \sigma(4) = 7\ ,\ \sigma(5) = 6\ ,\ \sigma(13) = 14\ ,\ \sigma(41) = 42\,.$$

From this list we see that the integers n with $\sigma(n) = 42$ are $n = 41$, $n = 13 \cdot 2 = 26$ and $n = 5 \cdot 4 = 20$.

(c) Find the smallest positive integer n such that $\tau(n) = 42$.

Solution: 42 can be factored as $42 = 2 \cdot 3 \cdot 7 = 2 \cdot 21 = 3 \cdot 14 = 7 \cdot 6 = 42$ and so the positive integers n with $\tau(n) = 42$ are of one of the forms $p^1q^2r^6$, p^1q^{20}, p^2q^{13} or p^{41} for some distinct primes p, q, r. The smallest such integer is $n = 2^6 \cdot 3^2 \cdot 5^1 = 2880$.

(d) For which positive integers n is $\tau(n)$ odd?

Solution: $\tau(n)$ is odd when all primes have an even exponent in the prime factorization of n, that is when n is a square.

(e) For which positive integers n is $\sigma(n)$ odd?

Solution: $\sigma(2^k) = (1 + 2 + 4 + \cdots + 2^k)$ is odd for all values of $k \geq 1$ and, for an odd prime p, $\sigma(p^k) = (1 + p + p^2 + \cdots + p^k)$ is odd when k is even. Thus $\sigma(n)$ is odd when all odd primes have an even exponent in the prime factorization of n, that is when n is either a square or twice a square.

3: Let $a = (25)!$ and $b = (5500)^3(1001)^2$.

(a) Find the prime factorization of a and of b.

Solution: Recall that the exponent of the prime p in $n!$ is $\lfloor \frac{n}{p} \rfloor + \lfloor \frac{a}{p^2} \rfloor + \lfloor \frac{a}{p^3} \rfloor$, so we have

$$\begin{aligned} a &= 2^{12+6+3+1} \cdot 3^{8+2} \cdot 5^{5+1} \cdot 7^3 \cdot 11^2 \cdot 13^1 \cdot 17^1 \cdot 19^1 \cdot 23^1 \\ &= 2^{22} \cdot 3^{10} \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13^1 \cdot 17^1 \cdot 19^1 \cdot 23^1 \\ b &= (2^2 \cdot 5^3 \cdot 11^1)^3 (7^1 \cdot 11^1 \cdot 13^1)^2 \\ &= 2^6 \cdot 5^9 \cdot 7^2 \cdot 11^5 \cdot 13^2 \,. \end{aligned}$$

(b) Find the prime factorization of $\gcd(a, b)$ and of $\operatorname{lcm}(a, b)$.

Solution: From the prime factorizations of a and b we obtain

$$\begin{aligned} \gcd(a, b) &= 2^6 \cdot 3^0 \cdot 5^6 \cdot 7^2 \cdot 11^2 \cdot 13^1 \\ \operatorname{lcm}(a, b) &= 2^{22} \cdot 3^{10} \cdot 5^9 \cdot 7^3 \cdot 11^5 \cdot 13^2 \cdot 17^1 \cdot 19^1 \cdot 23^1 \,. \end{aligned}$$

(c) Find the number of positive factors of b which are not factors of a.

Solution: Recall that the number $N(n)$ of positive factors of $n = {p_1}^{k_1} {p_2}^{k_2} \cdots {p_l}^{k_l}$ is given by the formula $N(n) = (k_1 + 1)(k_2 + 1) \cdots (k_l + 1)$. Let N be the number of positive factors of b which are not factors of a. Notice that $N = N(b) - N(d)$, where $d = \gcd(a, b)$, so by the factorizations from parts (a) and (b) we have

$$N = N(b) - N(d) = 7 \cdot 10 \cdot 3 \cdot 6 \cdot 3 - 7 \cdot 7 \cdot 3 \cdot 3 \cdot 2 = 2898 \,.$$

(d) Find the number of factors (positive or negative) of b which are either perfect squares or perfect cubes (or both).

Solution: Let $n = {p_1}^{k_1} {p_2}^{k_2} \cdots {p_l}^{k_l}$, then the positive factors of n which are perfect squares are of the form $r = {p_1}^{j_1} {p_2}^{j_2} \cdots {p_l}^{j_l}$ where each j_i is even with $0 \le j_i \le k_i$. For each i, the number of choices for j_i is $\lfloor \frac{k_i}{2} \rfloor + 1$, so the total number of positive factors which are perfect squares is $\left(\lfloor \frac{k_1}{2} \rfloor + 1\right)\left(\lfloor \frac{k_2}{2} \rfloor + 1\right) \cdots \left(\lfloor \frac{k_l}{2} \rfloor + 1\right)$. Similarly, the number $N(n, m)$ of positive factors of n which are perfect m^{th} powers is

$$N(n, m) = \left(\left\lfloor \frac{k_1}{m} \right\rfloor + 1 \right) \left(\left\lfloor \frac{k_2}{m} \right\rfloor + 1 \right) \cdots \left(\left\lfloor \frac{k_l}{m} \right\rfloor + 1 \right) .$$

Using this formula, we have $N(b, 2) = 4 \cdot 5 \cdot 2 \cdot 3 \cdot 2 = 240$, $N(b, 3) = 3 \cdot 4 \cdot 1 \cdot 2 \cdot 1 = 24$ and $N(b, 6) = 2 \cdot 2 \cdot 1 \cdot 1 \cdot 1 = 4$. Notice that perfect squares are always positive, but perfect squares can be positive or negative, so the total number of factors of b which are either perfect squares or perfect cubes is

$$N(b, 2) + 2N(b, 3) - N(b, 6) = 240 + 48 - 4 = 284 \,.$$

4: Solve the linear Diophantine equation $8x + 18y + 45z + 30w = 4$.

Solution: Since $\gcd(45, 30) = 15$, we can write $45z+30w = 15u$, that is $3z+2w = u$. Then the given equation reduces to $8x + 18y + 15u = 4$. Since $\gcd(18, 15) = 3$, we can write $18y + 15u = 3v$, that is $6y + 5u = v$. Then the given equation further reduces to $8x + 3v = 4$. By inspection, one solution to this reduced equation is $(x, v) = (-1, 4)$ and so the general solution is

$$(x, v) = (-1, 4) + k\,(-3, 8) = (-1 - 3k, 4 + 8k)\,.$$

Put $v = 4+8k$ back into the equation $6y+5u = v$ to get $6y+5u = 4+8k$. By inspection, one solution to the equation $6y + 5u = 1$ is $(y, u) = (1, -1)$, and so the general solution to the equation $6y + 5u = 4 + 8k$ is

$$(y, u) = (4 + 8k)(1, -1) + l(-5, 6) = (4 + 8k - 5l, -4 - 8k + 6l)\,.$$

Put $u = -4 - 8k + 6l$ back into the equation $3z + 2w = u$ to get $3z + 2w = -4 - 8k + 6l$. By inspection, one solution to the equation $3z + 2w = 1$ is $(z, w) = (1, -1)$, and so the general solution to the equation $3z + 2w = -4 - 8k + 6l$ is

$$(z, w) = (-4 - 8k + 6l)(1, -1) + m\,(-2, 3) = (-4 - 8k + 6l - 2m\,,\, 4 + 8k - 6l + 3m)\,.$$

Combining components of the above solutions (x, v), (y, u) and (z, w), we see that the solution to the original equation is

$$\begin{aligned}(x, y, z, w) &= (-1 - 3k\,,\, 4 + 8k - 5l\,,\, -4 - 8k + 6l - 2m\,,\, 4 + 8k - 6l + 3m)\\ &= (-1, 4, -4, 4) + k\,(-3, 8, -8, 8) + l\,(0, -5, 5, -6) + m\,(0, 0, -2, 3)\,,\end{aligned}$$

where $k, l, m \in \mathbf{Z}$. We remark that there are other correct ways to express this solution which use a different particular solution and a different basis of direction vectors, for example

$$(x, y, z, w) = (-2, 1, 0, 0) + k\,(3, -3, 0, 1) + l\,(0, -5, 2, 0) + m\,(0, 0, -2, 3)\,.$$

5: Consider the following system of linear Diophantine equations.

$$\begin{aligned} 5x + y + 4z + w &= a \\ 4y + 6z + 9w &= 2 \end{aligned}$$

(a) Find all integers a such that the system has a solution.

Solution: We solve the second equation $4y + 6z + 9w$. Since $\gcd(6, 9) = 3$ we can write $6z + 9w = 3u$, that is $2z + 3w = u$. Then the second equation reduces to $4y + 3y = 2$. By inpection, the solution to this reduced equation is

$$(y, u) = (-1, 2) + k\,(-3, 4) = (-1 - 3k\,,\, 2 + 4k)\,.$$

Put $u = 2 + 4k$ back into the equation $2z + 3w = u$ to get $2z + 3w = 2 + 4k$. By inspection, a solution to the equation $2z + 3w = 1$ is $(z, w) = (-1, 1)$, and so the general solution to the equation $2z + 3w = 2 + 4k$ is

$$(z, w) = (2 + 4k)(-1, 1) + l\,(-3, 2) = (-2 - 4k - 3l\,,\, 2 + 4k + 2l)\,.$$

Combining the above solutions (y, u) and (z, w) we find that the general solution to the second of the two given equations is

$$(y, z, w) = (-1 - 3k\,,\, -2 - 4k - 3l\,,\, 2 + 4k + 2l)\,.$$

Put this solution back into the first of the two given equations (that is the equation $5x + y + 4z + w = a$) to get $5x - 1 - 3k - 8 - 16k - 12l + 2 + 4k + 2l = a$, that is

$$5x - 15k - 10l = a + 7\,.$$

In order for a solution to exist, we need $\gcd(5, -15, -10)$ to divide $a + 7$, so $a + 7$ must be a multiple of 5. Thus a solution exists when a is of the form $a = 3 + 5r$ for some $r \in \mathbf{Z}$.

(b) Solve the system when $a = 3$.

Solution: When $a = 3$ the equation $5x - 15k - 10l = a + 7$ becomes $5x - 15k - 10l = 10$, or equivalently $x - 3k - 2l = 2$. Write $-3k - 2l = v$ so this equation becomes $x + v = 2$. By inspection, the solution to the equation $x + v = 2$ is

$$(x, v) = (1, 1) + s(-1, 1) = (1 - s\,,\, 1 + s)\,.$$

Put $v = 1 + s$ back into the equation $-3k - 2l = v$ to get $-3k - 2l = 1 + s$. The general solution to this is

$$(k, l) = (1 + s)(-1, 1) + t(2, -3) = (-1 - s + 2t, 1 + s - 3t)\,.$$

From our solution (x, v), and by substituting the above values for k and l back into our solution (y, z, w) from part (a), and we obtain the final solution:

$$\begin{aligned} (x, y, z, w) &= (1 - s, -1 - 3k\,,\, -2 - 4k - 3l\,,\, 2 + 4k + 2l) \\ &= (1 - s\,,\, -1 + 3 + 3s - 6t\,,\, -2 + 4 + 4s - 8t - 3 - 3s + 9t\,,\, 2 - 4 - 4s + 8t + 2 + 2s - 6t) \\ &= (1 - s, 2 + 3s - 6t\,,\, -1 + s + t\,,\, -2s + 2t) \\ &= (1, 2, -1, 0) + s\,(-1, 3, 1, -2) + t\,(0, -6, 1, 2)\,, \end{aligned}$$

where $s, t \in \mathbf{Z}$.