PMATH 340 Number Theory, Solutions to Assignment 4.5

1: (a) Express the (periodic) continued fraction $x=[1,3,\overline{1,1,2}\,]$ in the form $x=\frac{r-\sqrt{d}}{s}$ with $r,s,d\in\mathbb{Z}^+$.

Solution: Let $x=[1,3,\overline{1,1,2}\,]$ and let $u=[\,\overline{1,1,2}\,]$. Then

$$u=[1,1,2,u]=1+\frac{1}{1+\frac{1}{2+\frac{1}{u}}}=1+\frac{1}{1+\frac{u}{2u+1}}=1+\frac{2u+1}{3u+1}=\frac{5u+2}{3u+1}$$

so that $3u^2+u=5u+2$ hence $3u^2-4u-2=0$. Thus $u=\frac{4\pm\sqrt{16+24}}{6}=\frac{2\pm\sqrt{10}}{3}$. Since $u>1$ we must have $u=\frac{2+\sqrt{10}}{3}$. Thus

$$x=[1,3,\overline{1,1,2}]=[1,3,u]=1+\frac{1}{3+\frac{1}{u}}=1+\frac{u}{3u+1}=\frac{4u+1}{3u+1}=\frac{4\cdot\frac{2+\sqrt{10}}{3}+1}{3\cdot\frac{2+\sqrt{10}}{3}+1}$$

$$=\frac{1}{3}\cdot\frac{4(2+\sqrt{10})+3}{(2+\sqrt{10}+1}=\frac{1}{3}\cdot\frac{4\sqrt{10}+11}{\sqrt{10}+3}\cdot\frac{\sqrt{10}-3}{\sqrt{10}-3}=\frac{1}{3}\cdot\frac{7-\sqrt{10}}{1}=\frac{7-\sqrt{10}}{3}.$$

(b) Find the 4^{th} convergent $c_4=\frac{p_4}{q_4}$ for the continued fraction representation of e^2. You can use a calculator.

Solution: We list the terms x_k, a_k, p_k and q_k where $x_0=e^2$, $a_k=\lfloor x_k\rfloor$, $x_{k+1}=\frac{1}{x_k-a_k}$, $p_0=a_0$, $p_1=a_1a_0+1$, $p_k=a_kp_{k-1}+p_{k-2}$, $q_0=1$, $q_1=a_1$ and $q_k=a_{k-1}q_k+q_{k-2}$.

k	x_k	a_k	p_k	q_k
0	$e^2\cong 7.389$	7	7	1
1	$\frac{1}{e^2-7}\cong 2.570$	2	15	2
2	$\frac{1}{\frac{1}{e^2-7}-2}=\frac{e^2-7}{15-2e^2}\cong 1.753$	1	22	3
3	$\frac{1}{\frac{e^2-7}{15-2e^2}-1}=\frac{15-2e^2}{3e^2-22}\cong 1.327$	1	37	5
4	$\frac{1}{\frac{15-2e^2}{3e^2-22}-1}=\frac{3e^2-22}{37-5e^2}\cong 3.055$	3	133	18

The 4^{th} convergent is $c_4=\frac{p_4}{q_4}=\frac{133}{18}$ (which, as you can check, is quite close to e^2).

(c) Express $\sqrt{43}$ as a continued fraction and find the smallest unit $u>1$ in $\mathbb{Z}[\sqrt{43}\,]$.

Solution: We list the terms x_k, a_k, p_k and q_k where $x_0=\sqrt{43}$, $a_k=\lfloor x_k\rfloor$, $x_{k+1}=\frac{1}{x_k-a_k}$, $p_0=a_0$, $p_1=a_1a_0+1$, $p_k=a_kp_{k-1}+p_{k-2}$, $q_0=1$, $q_1=a_1$ and $q_k=a_kq_{k-1}+q_{k-2}$. We also include $N_k={p_k}^2-43\,{q_k}^2$.

k	x_k	a_k	p_k	q_k	N_k
0	$\sqrt{43}$	6	7	1	-7
1	$\frac{1}{\sqrt{43}-6}=\frac{\sqrt{43}+6}{7}$	1	7	1	6
2	$\frac{7}{\sqrt{43}-1}=\frac{\sqrt{43}+1}{6}$	1	13	2	-3
3	$\frac{6}{\sqrt{43}-5}=\frac{\sqrt{43}+5}{3}$	3	46	7	9
4	$\frac{3}{\sqrt{43}-4}=\frac{\sqrt{43}+4}{9}$	1	59	9	-2
5	$\frac{9}{\sqrt{43}-5}=\frac{\sqrt{43}+5}{2}$	5	341	52	9
6	$\frac{2}{\sqrt{43}-5}=\frac{\sqrt{43}+5}{9}$	1	400	61	-3
7	$\frac{9}{\sqrt{43}-4}=\frac{\sqrt{43}+4}{3}$	3	1541	235	6
8	$\frac{3}{\sqrt{43}-5}=\frac{\sqrt{43}+5}{6}$	1	1941	296	-7
9	$\frac{6}{\sqrt{43}-1}=\frac{\sqrt{43}+1}{7}$	1	3482	531	1
10	$\frac{7}{\sqrt{43}-6}=\frac{\sqrt{43}+6}{1}$	12			

From the table, the continued fraction for $\sqrt{43}$ is $\sqrt{43}=\big[\,6,\,\overline{1,1,3,1,5,1,3,1,1,12}\,\big]$ and the smallest unit $u>1$ in $\mathbb{Z}[\sqrt{43}\,]$ is $u=p_9+q_9\sqrt{43}=3482+531\sqrt{43}$.

2: (a) Find all solutions $(x,y)\in\mathbb{Z}^2$ to Pell's Equation $x^2-29\,y^2=1$.

Solution: The following table lists the data used to calculate the continued fraction for $\sqrt{29}$ and the first few convergents $c_k=\frac{p_k}{q_k}$ along with the norms $N_k=N(p_k+q_k\sqrt{29})=p_k^2-29\,q_k^2$.

k	x_k	a_k	p_k	q_k	N_k
0	$\sqrt{29}$	5	5	1	-4
1	$\frac{1}{\sqrt{29}-5}=\frac{\sqrt{29}+5}{4}$	2	11	2	5
2	$\frac{4}{\sqrt{29}-3}=\frac{\sqrt{29}+3}{5}$	1	16	3	-5
3	$\frac{5}{\sqrt{29}-2}=\frac{\sqrt{29}+2}{5}$	1	27	5	4
4	$\frac{5}{\sqrt{29}-3}=\frac{\sqrt{29}+3}{4}$	2	70	13	-1
5	$\frac{4}{\sqrt{29}-5}=\frac{\sqrt{29}+5}{1}$	10			

We have $\sqrt{29}=[5,\overline{2,1,1,2,10}]$ with period $\ell=5$. Writing $u_k=p_k+q_k\sqrt{29}\in\mathbb{Z}[\sqrt{29}\,]$, the smallest unit in $\mathbb{Z}[\sqrt{29}\,]$ with $u>1$ is $u=u_{\ell-1}=u_4=70+13\sqrt{29}$, and we have $N(u)=-1$. The smallest unit v in $\mathbb{Z}[\sqrt{29}\,]$ with $v>1$ and $N(v)=1$ is

$$v=u^2=(70+13\sqrt{29})^2=9801+1820\sqrt{29}.$$

If we write $v^k=(9801+1820\sqrt{29})^k=r_k+s_k\sqrt{29}$ for $0\le k\in\mathbb{Z}$, then the solutions to Pell's equation $x^2-29y^2=1$ are given by $(x,y)=(\pm r_k,\pm s_k)$ where $0\le k\in\mathbb{Z}$. We also remark that it is not hard to give a recursion formula for p_k and q_k, and it is also possible to give an ugly closed-form formula for p_k and q_k.

(b) Find all solutions $(x,y)\in\mathbb{Z}^2$ to the Pell-like equation $x^2-21\,y^2=4$.

Solution: The following table lists the data used to calculate the continued fraction for $\sqrt{21}$ and the first few convergents $c_k=\frac{p_k}{q_k}$ along with the norms $N_k=N(p_k+q_k\sqrt{21})=p_k^2-21\,q_k^2$.

k	x_k	a_k	p_k	q_k	N_k
0	$\sqrt{21}$	4	4	1	-5
1	$\frac{1}{\sqrt{21}-4}=\frac{\sqrt{21}+4}{5}$	1	5	1	4
2	$\frac{5}{\sqrt{21}-1}=\frac{\sqrt{21}+1}{4}$	1	9	2	-3
3	$\frac{4}{\sqrt{21}-3}=\frac{\sqrt{21}+3}{3}$	2	23	5	4
4	$\frac{3}{\sqrt{21}-3}=\frac{\sqrt{21}+3}{4}$	1	32	7	-5
5	$\frac{4}{\sqrt{21}-1}=\frac{\sqrt{21}+1}{5}$	1	55	12	1
6	$\frac{5}{\sqrt{21}-4}=\frac{\sqrt{21}+4}{1}$	8			

We have $\sqrt{21}=[4,\overline{1,1,2,1,1,8}]$ with period $\ell=6$. Writing $u_k=p_k+q_k\sqrt{21}\in\mathbb{Z}[\sqrt{21}\,]$, the smallest unit in $\mathbb{Z}[\sqrt{21}\,]$ with $u>1$ is $u=u_{\ell-1}=u_5=55+12\sqrt{21}$, and we have $N(u)=1$.

If (x,y) is a solution with $x,y\in\mathbb{Z}^+$ (so we have $x^2-21y^2=4$) then since $4\le\sqrt{21}$ it follows, from Corollary 7.12, that $\frac{x}{y}$ is equal to one of the convergents $\frac{p_k}{q_k}$ of $\sqrt{21}$. In order to have $\frac{x}{y}=\frac{p_k}{q_k}$ we need $x=tp_k$ and $y=tq_k$ for some $t\in\mathbb{Z}^+$ and then we have $N_k={p_k}^2-21{q_k}^2=\frac{x^2-21y^2}{t^2}=\frac{4}{t^2}$, and it follows that $t\in\{1,2\}$. When $t=1$ so that $x=p_k$ and $y=q_k$ we have $N_k=4$ and when $t=2$ so that $x=2p_k$ and $y=2p_k$ we have $N_k=1$. Since the sequence of norms N_k is periodic, with $N_k=4$ for $k=1,3\text{mod } 6$ and $N_k=1$ for $k=5\text{mod } 6$, all of the positive solutions (x,y) are given by $(x,y)=(p_k,q_k)$ for $k=1,3\text{mod } 6$ and $(x,y)=(2p_k,2q_k)$ for $k=5\text{mod } 6$. If we write $v=u_1=5+\sqrt{21}$ and $w=u_3=23+5\sqrt{21}$ then since $N(u)=1$ so that $N(u^k)=1$, $N(vu^k)=N(v)=4$ and $N(wu^k)=N(w)=4$ for all $k\ge0$, the solutions (x,y) with $x,y\in\mathbb{Z}^+$ correspond to the elements $x+y\sqrt{21}\in\mathbb{Z}[\sqrt{21}]$ in the following increasing sequence

$$v,w,2u,vu,wu,2u^2,vu^2,wu^2,2u^3,\cdots$$

We can simplify our description of the solutions if we notice that $\left(\frac{v}{2}\right)^2=\left(\frac{5+\sqrt{21}}{2}\right)^2=\frac{23+5\sqrt{21}}{2}=\frac{w}{2}$ and $\left(\frac{v}{2}\right)^3=\frac{5+\sqrt{21}}{2}\cdot\frac{23+5\sqrt{21}}{2}=55+12\sqrt{21}=u$ so the above sequence $v,w,2u,vu,\cdots$ can be written as $2\left(\frac{v}{2}\right)^k$, $k\in\mathbb{Z}^+$. Thus the solutions are given by $(x,y)=(\pm r_k,\pm s_k)$ where $r_k+s_k\sqrt{21}=2\left(\frac{5+\sqrt{21}}{2}\right)^k$ for $0\le k\in\mathbb{Z}$.