PMATH 340 Number Theory, Assignment 1 Due: Mon Feb 5, 11:00 pm

1: (a) Solve the Linear Diophantine Equation $385x - 1183y = 294$.

(b) A shopper spends \$19.81 to buy some bananas which cost 35 cents each and some plums which cost 56 cents each. What is the minimum number of pieces of fruit that the shopper could have bought.

2: We can solve a pair of linear diophantine equations in three variables by first eliminating one of the variables and solving the resulting equation in the remaining two variables.

(a) Show that there is no solution to the pair of diophantine equations

$$\begin{aligned} 2x + 7y + z &= 45 \qquad (1) \\ 3x + 8y + 4z &= 21 \qquad (2) \end{aligned}$$

(b) Find all solutions to the pair of diophantine equations

$$\begin{aligned} 20x + 12y + 15z &= 85 \qquad (1) \\ 18x + 20y + 8z &= 110 \qquad (2) \end{aligned}$$

3: (a) Find the prime factorization of $n = 2^{36} - 1$.

(b) Find the exponent of 3 in the prime factorization of $\binom{100}{40} = \frac{(100)!}{(40)!(60)!}$.

(c) Let $a = \prod_{k=1}^{6} k^k$. Find the number of factors (positive or negative) of a which are either perfect squares or perfect cubes (or both).

4: (a) Prove that for all positive integers a and b, we have $a \big| b$ if and only if $a^2 \big| b^2$.

(b) Prove that for all positive integers a, b and c, if $c \big| ab$ then $c \big| \gcd(a,c)\gcd(b,c)$.

(c) Prove that for all positive integers a, b and c, we have $\gcd(ac, bc) = c \gcd(a,b)$.

(d) Prove that for all positive integers a and b, we have $\gcd(a,b) = \gcd\big(a+b, \operatorname{lcm}(a,b)\big)$.