PMATH 333 Introduction to Real Analysis, Solutions to the Term Test, Winter 2024

[10] **1:** (a) Let R be a ring. Using only the rules R0-R7, with one rule used at each step, prove that for all $a, b \in R$, if $(a+b)\,x = x + b$ for every $x \in R$, then we have $b = 0$ and $a = 1$.

Solution: Let $a, b \in R$. Suppose that $(a+b)\cdot x = x+b$ for all $x \in R$. Then, in particular (taking $x = 0$) we have $(a+b)\cdot 0 = 0 + b$ and so

$$\begin{aligned} 0 &= (a+b)\cdot 0 \text{ , by R0,} \\ &= 0 + b \text{ , since } (a+b)\cdot 0 = 0+b, \\ &= b + 0 \text{ , by R2,} \\ &= b \text{ , by R3.} \end{aligned}$$

This proves that $b = 0$. Since $(a+b)\cdot x = x+b$ for all $x \in R$ it follows in particular, by taking $x = 1$, that $(a+b)\cdot 1 = 1+b$ and, since $b = 0$, we have $(a+0)\cdot 1 = 1+0$. Thus

$$\begin{aligned} a &= a + 0 \text{ , by R3,} \\ &= (a+0)\cdot 1 \text{ , by R6,} \\ &= 1 + 0 \text{ , since } (a+0)\cdot 1 = 1+0, \\ &= 1 \text{ , by R3.} \end{aligned}$$

(b) Prove the floor property of $\mathbb{Z}$ in $\mathbb{R}$: for every $x \in \mathbb{R}$ there exists a unique $n \in \mathbb{Z}$ with $x - 1 < n \le x$.

Solution: First we prove uniqueness. Let $x \in \mathbb{R}$ and suppose that $n, m \in \mathbb{Z}$ with $x - 1 < n \le x$ and $x - 1 < m \le x$. Since $x - 1 < n$ we have $x < n + 1$. Since $m \le x$ and $x < n+1$ we have $m < n+1$ hence $m \le n$. Similarly, we have $n \le m$. Since $n \le m$ and $m \le n$, we have $n = m$. This proves uniqueness.

Next we prove existence. Let $x \in \mathbb{R}$. First let us consider the case that $x \ge 0$. Let $A = \{k \in \mathbb{Z} \,|\, k \le x\}$. Note that $A \ne \emptyset$ because $0 \in A$ and A is bounded above in $\mathbb{R}$ by x. By The Well-Ordering Property of $\mathbb{Z}$ in $\mathbb{R}$, A has a maximum element. Let $n = \max A$. Since $n \in A$ we have $n \in \mathbb{Z}$ and $n \le x$. Also note that $x - 1 < n$ since $x - 1 \ge n \Longrightarrow x \ge n+1 \Longrightarrow n+1 \in A \Longrightarrow n \ne \max A$. Thus for $n = \max A$ we have $n \in \mathbb{Z}$ with $x - 1 < n \le x$, as required.

Next consider the case that $x < 0$. If $x \in \mathbb{Z}$ we can take $n = x$. Suppose that $x \notin \mathbb{Z}$. We have $-x > 0$ so, by the previous paragraph, we can choose $m \in \mathbb{Z}$ with $-x - 1 < m \le -x$. Since $m \in \mathbb{Z}$ but $x \notin \mathbb{Z}$ we have $m \ne -x$ so that $-x-1 < m < -x$ and hence $x < -m < x+1$. Thus we can take $n = -m-1$ to get $x - 1 < n < x$. This completes the proof of Part (1).

[10] **2:** (a) Let $x_n = \frac{n^2+1}{(2n+1)^2}$ for $n \geq 0$. Use the definition of the limit to prove that $\lim\limits_{n\to\infty} x_n = \frac{1}{4}$.

Solution: Note that for $n \geq 1$ we have

$$\left|x_n - \tfrac{1}{4}\right| = \left|\tfrac{n^2+1}{(2n+1)^2} - \tfrac{1}{4}\right| = \left|\tfrac{4n^2+4-(4n^2+4n+1)}{4(2n+1)^2}\right| = \left|\tfrac{3-4n}{4(2n+1)^2}\right| = \tfrac{4n-3}{4(2n+1)^2} \leq \tfrac{4n}{4(2n)^2} = \tfrac{1}{4n}.$$

Let $\epsilon > 0$. Choose $m \in \mathbb{Z}^+$ so that $\frac{1}{4m} < \epsilon$. Then for $n \geq m$, as shown above we have

$$\left|x_n - \tfrac{1}{4}\right| \leq \tfrac{1}{4n} \leq \tfrac{1}{4m} < \epsilon.$$

Thus $\lim\limits_{n\to\infty} x_n = \frac{1}{4}$, as required.

(b) Prove the following part of the Extreme Value Theorem: if $f\colon [a,b] \to \mathbb{R}$ is continuous then f attains its maximum value on $[a,b]$.

Solution: First we claim that f is bounded above. Suppose, for a contradiction, that it is not. For each $k \in \mathbb{Z}^+$, choose $x_k \in [a,b]$ such that $f(x_k) \geq k$. By the Bolzano Weierstrass Theorem, we can choose a convergent subsequence (x_{k_j}). Let $p = \lim\limits_{j\to\infty} x_{k_j}$. Note that $p \in [a,b]$ by Comparison (since $x_{k_j} \geq a$ for all j we have $p \geq a$, and since $x_{k_j} \leq b$ for all j we have $p \leq b$). Since $f(x_{k_j}) \geq k_j$ and $k_j \to \infty$ we must have $f(x_{k_j}) \to \infty$ as $j \to \infty$. But by the Sequential Characterization of Continuity, we should have $f(x_{k_j}) \to f(p) \in \mathbb{R}$, so we have obtained the desired contradiction. Thus f is bounded above, as claimed.

Since the range $f([a,b])$ is nonempty and bounded above, it has a supremum. Let $m = \sup f([a,b])$. By the Approximation Property of the supremum, for each $k \in \mathbb{Z}^+$ we can choose $y_k \in [a,b]$ such that $m - \frac{1}{k} \leq f(y_k) \leq m$. By the Bolzano Weierstrass Theorem, we can choose a convergent subsequence (y_{k_j}). Let $c = \lim\limits_{j\to\infty} y_{k_j}$. Since we have $m - \frac{1}{k_j} \leq f(y_{k_j}) \leq m$ and $\frac{1}{k_j} \to 0$, we have $f(y_{k_j}) \to m$ as $j \to \infty$ by the Squeeze Theorem. Since f is continuous at c, by the Sequential Characterization of Continuity we have $f(y_{k_j}) \to f(c)$, and so by the Uniqueness of Limits, we have $f(c) = m$. Thus f attains its maximum value at c.

[10] **3:** (a) Let $f\colon [a,b]\to\mathbb{R}$ be continuous. Prove that there exist $r,s\in\mathbb{R}$ with $r\le s$ such that $\text{Range}(f)=[r,s]$.

Solution: By the Extreme Value Theorem, since f is continuous, we can choose $c\in[a,b]$ such that $f(c)\le f(x)$ for all $x\in[a,b]$ and we can choose $d\in[a,b]$ such that $f(d)\ge f(x)$ for all $x\in[a,b]$. Let $r=f(c)$ and let $s=f(d)$. Then we have $r=f(c)\le f(x)\le f(d)=s$ for all $x\in[a,b]$ and so $\big\{f(x)\,\big|\,x\in[a,b]\big\}\subseteq[r,s]$.

Let $y\in[r,s]$. Since $f(c)=r\le y\le s=f(d)$, and since f is continuous, it follows from the Intermediate Value Theorem that we can choose x between c and d (if $c\le d$ we can choose $x\in[c,d]$ and if $d\le c$ we can choose $x\in[d,c]$) such that $f(x)=y$. Thus we also have $[r,s]\subseteq\big\{f(x)\,\big|\,x\in[a,b]\big\}$.

(b) Let $f\colon [a,b)\to\mathbb{R}$. Prove that if f is uniformly continuous on $[a,b)$ then f is bounded.

Solution: Since f is uniformly continuous, we can choose $\delta>0$ such that for all $x,y\in[a,b)$ with $|x-y|<\delta$ we have $|f(x)-f(y)|<1$. Choose $n\in\mathbb{Z}^+$ such that $\frac{b-a}{n}\le\delta$. Let $X=\{x_0,x_1,\cdots,x_n\}$ be the partition of $[a,b]$ into n-equal-sized subintervals, so we have $x_k=a+\frac{b-a}{n}\,k$. Let $c\in[a,b)$. Let ℓ be the integer with $1\le\ell\le n$ such that $c\in\big[x_{\ell-1},x_\ell\big)$. Since $|x_k-x_{k-1}|<\delta$ and $|c-x_{\ell-1}|<\delta$, we have $\big|f(x_k)-f(x_{k-1})\big|\le 1$ and $|f(c)-f(x_{\ell-1})|\le 1$, and hence

$$\begin{aligned}|f(c)|=\big|f(a)+f(c)-f(a)\big|&=\Big|f(a)+\big(f(c)-f(x_{\ell-1})\big)+\sum_{k=1}^{\ell-1}\big(f(x_k)-f(x_{k-1})\big)\Big|\\&\le|f(a)|+|f(c)-f(x_{\ell-1})|+\sum_{k=1}^{\ell-1}|f(x_k)-f(x_{k-1})|\\&\le|f(a)|+1+(\ell-1)=|f(a)|+\ell\le|f(a)|+n.\end{aligned}$$

Thus f is bounded with $|f(c)|\le|f(a)|+n$ for all $c\in[a,b)$.

[10] **4:** (a) Let $f\colon [a,b]\subseteq\mathbb{R}\to\mathbb{R}$ be increasing. Prove that f is integrable.

Solution: Suppose that f is increasing $\big($and hence bounded, below by $f(a)$ and above by $f(b)\big)$ on $[a,b]$. Let $\epsilon>0$. Choose a partition $X=\{x_0,x_1,\cdots,x_n\}$ of $[a,b]$ with $|X|<\frac{\epsilon}{f(b)-f(a)}$. Since f is increasing we have $f(x_k)=M_k=\max\big\{f(t)\,\big|\,t\in[x_{k-1},x_k]\big\}$ and $f(x_{k-1})=m_k=\min\big\{f(t)\,\big|\,t\in[x_{k-1},x_k]\big\}$, and so

$$\begin{aligned} U(f,X)-L(f,X) &= \sum_{k=1}^{n}(M_k-m_k)\Delta_k x = \sum_{k=1}^{n}\big(f(x_k)-f(x_{k-1})\big)\Delta_k x \\ &\le \sum_{k=1}^{n}\big(f(x_k)-f(x_{k-1})\big)|X| = \big(f(b)-f(a)\big)|X| < \epsilon. \end{aligned}$$

Thus f is integrable on $[a,b]$ (by the second Equivalent Definition of Integrability).

(b) Prove the following part of the Equivalent Definitions of Integrability Theorem: if $f\colon [a,b]\subseteq\mathbb{R}\to\mathbb{R}$ is integrable then $U(f)=L(f)$.

Solution: Suppose that f is integrable on $[a,b]$ with $I=\int_a^b f$. Let $\epsilon>0$. Choose $\delta>0$ so that for every partition X with $|X|<\delta$ we have $|S-I|<\frac{\epsilon}{4}$ for every Riemann sum S on X. Let X be a partition with $|X|<\delta$. Let S_1 be a Riemann sum for f on X with $|U(f,X)-S_1|<\frac{\epsilon}{4}$, and let S_2 be a Riemann sum for f on X with $|S_2-L(f,X)|<\frac{\epsilon}{4}$. Then

$$\begin{aligned} U(f,X)-L(f,X) &\le |U(f,X)-S_1|+|S_1-I|+|I-S_2|+|S_2-L(f,X)| \\ &< \tfrac{\epsilon}{4}+\tfrac{\epsilon}{4}+\tfrac{\epsilon}{4}+\tfrac{\epsilon}{4}=\epsilon\,. \end{aligned}$$

Since $U(f)\le U(f,X)$ and $L(f)\ge L(f,X)$, it follows that

$$U(f)-L(f)\le U(f,X)-L(f,X)<\epsilon.$$

Since $U(f)-L(f)<\epsilon$ for every $\epsilon>0$, we have $U(f)=L(f)$.