PMATH 333, Exercises for Chapter 1

1: Let R be a ring and let F be a field.

(a) Using only the rules R1-R9 which define a field, prove that for all $a \in F$ if $a \cdot a = a$ then ($a = 0$ or $a = 1$).

(b) Using only the rules R1-R9, prove that for all $a \in F$ if $a \cdot a = 1$ then ($a = 1$ or $a + 1 = 0$).

(c) Using only the rules R1-R7 which define a ring, together with the rule R0 which states that for all $a \in R$ we have ($a \cdot 0 = 0$ and $0 \cdot a = 0$), prove that for all $a, b, c, d \in R$, if $a + c = 0$ and $b + d = 0$ then $ab = cd$.

2: Let S be an ordered set and let F be an ordered field.

(a) Using only the rules O1-O3, and the rule O0 which defines the strict order $<$ by stating that for all $a, b \in S$ we have $a < b \iff (a \leq b \text{ and } a \neq b)$, prove that for all $a, b, c \in S$, if $a \leq b$ and $b < c$ then $a < c$.

(b) Using only the rules R1-R9 and O1-O5, prove that for all $a, b \in F$ if $0 \leq a$ and $a \leq b$ then $a \cdot a \leq b \cdot b$.

(c) Using only rules R1-R9 and O1-O5, together with the rule R0 from Exercise 1(c), prove that $0 \leq 1$.

3: In this problem, you may use any of the algebraic properties and order properties of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ described in Chapter 1 of the Lecture Notes.

(a) Let $A = \left\{ (-1)^n + \frac{1}{n} \,\middle|\, n \in \mathbb{Z}^+ \right\}$. Find (with proof) $\sup A$ and $\inf A$.

(b) Prove that for every $0 \leq y \in \mathbb{R}$ there exists a unique $0 \leq x \in \mathbb{R}$ such that $x^2 = y$ (this number x is called the *square root* of y and is denoted by $x = \sqrt{y} = y^{1/2}$). In other words, prove that the function $f : [0, \infty) \to [0, \infty)$ given by $f(x) = x^2$ is bijective.

4: In this problem, and in the following problem, you may use any known properties of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$.

(a) Let X and Y be nonempty sets and let $f : X \to Y$. Prove that f is injective if and only if we have $f(A \cap B) = f(A) \cap f(B)$ for all subsets $A, B \subseteq X$.

(b) Show that $|\mathbb{R}| = \big|[0, 1)\big|$ without using the Cantor-Schröder-Bernstein Theorem.

5: (a) Show that the cardinality of the set of all finite subsets of $\mathbb{N}$ is equal to $\aleph_0$.

(b) Show that the cardinality of the set of all functions from $\mathbb{N}$ to $\mathbb{N}$ is equal to $2^{\aleph_0}$.