PMATH 333, Solutions to the Exercises for Appendix 2

1: Recall that a formula in first-order set theory only uses symbols from the following symbol set:

$$\neg\,,\,\wedge\,,\,\vee\,,\;\rightarrow\;,\;\leftrightarrow\;,\,(\,,\,)\,,\,=\,,\,\in\,,\,\exists\,,\,\forall$$

along with variable symbols including $x, y, z, u, v, w, \cdots$.

(a) Express the statement $\{\emptyset, \{u\}\} \in w$ as a formula in first-order set theory.

Solution: In the class of sets we have

$$\begin{aligned} v=\{\emptyset,\{u\}\} &\iff \forall x\,\big(x\in v \leftrightarrow (x=\emptyset \vee x=\{u\})\big) \\ &\iff \forall x\big(x\in v \leftrightarrow (\forall y\,\neg\, y\in x \vee \forall y\,(y\in x \leftrightarrow y=u))\big) \end{aligned}$$

and so

$$\begin{aligned} \{\emptyset,\{u\}\}\in w &\iff \forall v\,\big(v=\{\emptyset,\{u\}\} \rightarrow v\in w\big) \\ &\iff \forall v\,\Big(\forall x\big(x\in v \leftrightarrow (\forall y\,\neg\, y\in x \vee \forall y\,(y\in x \leftrightarrow y=u))\big) \rightarrow v\in w\Big). \end{aligned}$$

Alternatively,

$$\begin{aligned} \{\emptyset,\{u\}\}\in w &\iff \exists v\,\big(v=\{\emptyset,\{u\}\} \wedge v\in w\big) \\ &\iff \exists v\,\Big(\forall x\big(x\in v \leftrightarrow (\forall y\,\neg\, y\in x \vee \forall y\,(y\in x \leftrightarrow y=u))\big) \wedge v\in w\Big). \end{aligned}$$

(b) Recall that $(x,y)=\big\{\{x\},\{x,y\}\big\}$. Express the statement "w is a set of ordered pairs" as a formula in first-order set theory.

Solution: In the class of all sets we have

$$\begin{aligned} w \text{ is a set of ordered pairs} &\iff \forall u\big(u\in w \rightarrow \exists x\,\exists y\; u=(x,y)\big) \\ &\iff \forall u\Big(u\in w \rightarrow \exists x\,\exists y\; u=\big\{\{x\},\{x,y\}\big\}\Big) \\ &\iff \forall u\Big(u\in w \rightarrow \exists x\,\exists y\,\forall v\big(v\in u \leftrightarrow (v=\{x\} \vee v=\{x,y\})\big)\Big) \\ &\iff \forall u\bigg(u\in w \rightarrow \exists x\,\exists y\,\forall v\Big(v\in u \leftrightarrow \big(\forall z(z\in v \leftrightarrow z=x) \vee \forall z(z\in v \leftrightarrow (z=x \vee z=y))\big)\Big)\bigg) \end{aligned}$$

We remark that the statement "w is a set of ordered pairs" is equivalent to the statement "w is a relation".

(c) Express the statement "for every $u\in w$ there exists $x\in u$ such that $u\setminus\{x\}\in w$" as a formula in first order set theory. Also, determine whether there exists such a set w which is not empty.

Solution: The given statement can be expressed as "$\forall u\!\in\! w\;\exists x\!\in\! u\;\forall y\big(y=u\setminus\{x\} \rightarrow y\in w\big)$" which, in turn, can be expressed by the formula

$$\forall u\Big(u\in w \rightarrow \exists x\Big(x\in u \wedge \forall y\Big(\forall z(z\in y \leftrightarrow (z\in u \wedge \neg\, z=x)) \rightarrow y\in w\Big)\Big)\Big)$$

There do exist such sets w, for example we could take w to be the set of all infinite subsets of $\mathbb{N}$, that is $w=\big\{u\in\mathcal{P}(\mathbb{N})\big|\exists n\!\in\!\mathbb{N}\; u\subseteq n\big\}$ which is a set by a Separation Axiom $\big($since $\mathcal{P}(\mathbb{N})$ is a set and the statement "$\exists n\!\in\!\mathbb{N}\; u\subseteq n$" can be expressed as a formula in First Order Set Theory$\big)$. As another example, we could take w to be the set $w=\big\{\{1,2,3,\cdots\},\{2,3,4,\cdots\},\{3,4,5,\cdots\},\cdots\big\}=\big\{u\in\mathcal{P}(\mathbb{N})\big|\exists n\!\in\!\mathbb{N}\;\forall x(x\in u \leftrightarrow n\in x\big\}$ which is a set by a Separation Axiom.

2: Recall that $\mathbb{N} = \{0, 1, 2, \cdots\}$ is a set where $0 = \emptyset$, $1 = \{0\}$, $2 = \{0, 1\}$ and in general $x + 1 = x \cup \{x\}$.

(a) Show that if u is a set then the collection $w = \big\{x \cup \{x\} \,\big|\, x \in u\big\}$ is a set.

Solution: We provide two solutions by explaining how w can be constructed from u using the ZFC axioms in two slightly different ways. In both solutions we use the fact that the statement "$y = x \cup \{x\}$" is considered to be an allowable mathematical statement because it can be expressed as the first-order formula

$$F(x, y) \equiv \forall z\big(z \in y \leftrightarrow (z \in x \vee z = x)\big) .$$

For the first solution, we note that when u is a set, the given collection $w = \big\{x \cup \{x\} \big| x \in u\big\}$ is equal to

$$w = \big\{y \,\big|\, \exists x \in u \;\, y = x \cup \{x\}\big\} = \big\{y \,\big|\, \exists x \in u \;\, F(x, y)\big\} ,$$

which is a set by a Replacement Axiom, because the statement $F(x, y)$ has the property that for every set x there is a unique set y such that $F(x, y)$ is true (indeed, given a set x, to make $F(x, y)$ true we must take $y = x \cup \{x\}$, which is a set by the Pair and Union Axioms).

For the second solution, note that when x, y and u are sets with $x \in u$, if $y \in x$ then $y \in \bigcup u$ and if $y \in \{x\}$ then $y = x$ so $y \in u$, and so if $y \in x \cup \{x\}$ then $y \in u \cup \bigcup u$. Thus the given collection is

$$w = \big\{x \cup \{x\} \big| x \in u\big\} = \big\{y \,\big|\, \exists x \in u \;\, F(x, y)\big\} = \Big\{y \in u \cup \bigcup u \,\Big|\, \exists x (x \in u \wedge F(x, y))\Big\}$$

which is a set by a Separation Axiom, since $u \cup \bigcup u$ is a set by the Pair and/or Union Axioms.

(b) Show that the collection $w = \big\{\{0, 1\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \cdots\big\}$ is a set.

Solution: Again we provide two solutions. Note that when x and y are sets we have

$$\begin{aligned} y = \big\{x, x \cup \{x\}\big\} \;&\Longleftrightarrow\; \forall z\big(z \in y \leftrightarrow (z = x \vee z = x \cup \{x\})\big) \\ &\Longleftrightarrow\; \forall z\Big(z \in y \leftrightarrow \big(z = x \vee \forall u (u \in z \leftrightarrow (u \in x \vee u = x))\big)\Big) \end{aligned}$$

so the statement "$y = \big\{x, x \cup \{x\}\big\}$" can be expressed as the formula

$$F(x, y) \equiv \forall z\Big(z \in y \leftrightarrow \big(z = x \vee \forall u (u \in z \leftrightarrow (u \in x \vee u = x))\big)\Big) .$$

For the first solution we note that

$$\begin{aligned} w &= \big\{\{0, 1\}, \{1, 2\}, \{2, 3\}, \cdots\big\} = \big\{\{x, x + 1\} \big| x \in \mathbb{N}\big\} = \big\{\{x, x \cup \{x\}\} \big| x \in \mathbb{N}\big\} \\ &= \big\{y \,\big|\, \exists x \in \mathbb{N} \;\, F(x, y)\big\} \end{aligned}$$

which is a set by a Replacement Axiom, since $\mathbb{N}$ is known to be a set by the Axiom of Infinity, and since the statement $F(x, y)$ has the property that for every set x there exists a unique set y such that the statement is true (indeed given x, to make the statement true we must choose $y = \big\{x, x \cup \{x\}\big\}$, which is a set by the Pair and Union Axioms).

For the second solution, note that when $x \in \mathbb{N}$ we have $\{x, x + 1\} \in P(\mathbb{N})$ and so

$$w = \big\{\{x, x + 1\} \big| x \in \mathbb{N}\big\} = \big\{y \,\big|\, \exists x \in \mathbb{N} \;\, y = \{x, x + 1\}\big\} = \Big\{y \in P(\mathbb{N}) \,\Big|\, \exists x\big(x \in \mathbb{N} \wedge F(x, y)\big)\Big\}$$

which is a set by a Separation Axiom, since $P(\mathbb{N})$ is a set by the Axiom of Infinity and the Power Set Axiom, and since the statement "$x \in \mathbb{N} \,\wedge\, F(x, y)$" is an allowable mathematical statement.

To be careful, we should verify that the statements "$u = \mathbb{N}$" and "$x \in \mathbb{N}$" can be expressed as formulas (in first-order set theory) so that they are allowable mathematical statements. It is not clear, from reading Chapter 1 in the Lecture Notes, exactly how this should be done. After reading the definition of the set $\mathbb{N}$ given in Appendix 1, you will be able to work out that the statement "$u = \mathbb{N}$" can be expressed as

$$\emptyset \in u \wedge \forall x (x \in u \to x \cup \{x\} \in u) \wedge \forall w\Big(\big(\emptyset \in w \wedge \forall x (x \in w \to x \cup \{x\} \in w)\big) \to u \subseteq w\Big)$$

which can, in turn, be expressed as a formula. The statement "$x \in \mathbb{N}$" can be expressed as $\forall u (u = \mathbb{N} \to x \in u)$.

(c) Show that the collection $w = \big\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \cdots\big\}$ is a set.

Solution: We shall only provide an incomplete solution. Define sets s_n, for $n \in \mathbb{N}$, recursively by $s_0 = \emptyset$ and $s_{n+1} = \{s_n\}$. If we can express the statement "$u = s_n$" as a formula $F(n, u)$ (with free variables n and u) then we have

$$w = \big\{s_n \big| n \in \mathbb{N}\big\} = \big\{u \,\big|\, \exists n{\in}\mathbb{N}\ u = s_n\big\} = \big\{u \,\big|\, \exists n{\in}\mathbb{N}\ F(u, n)\big\}$$

which is a set by a Replacement Axiom. For $n \in \mathbb{N}$, let

$$w_n = \big\{(0, \emptyset), (1, \{\emptyset\}), \cdots, (n, s_n)\big\}.$$

In order to show that the statement "$u = s_n$" is expressible as a formula, we shall first show that the (apparently more complicated) statement "$v = w_n$" is expressible as a formula. We recall that the statements "$u{=}\mathbb{N}$" and "$x{\in}\mathbb{N}$" are each expressible as formulas. When $n \in \mathbb{N}$ and v is a set we have

$$\begin{aligned} v = w_n \iff\ & \forall z{\in}v\ \exists x{\in}\{0, 1, \cdots, n\}\ \exists y\ z{=}(x, y) \\ \text{and}\ \ & \forall x{\in}\{0, 1, \cdots, n\}\ \exists y\ (x, y){\in}v \\ \text{and}\ \ & \forall x\ \forall y\ \forall z\ \big(\big((x, y){\in}v \wedge (x, z){\in}v\big) \to y{=}z\big)\big) \\ \text{and}\ \ & (0, \emptyset) \in v \\ \text{and}\ \ & \forall x{\in}\{0, 1, \cdots, n{-}1\}\ \forall y\ \big((x, y){\in}v \to (x{+}1, \{y\}){\in}v\big) \end{aligned}$$

We leave it as a (long but not particularly difficult) exercise to verify that the rather long statement on the right can be expressed as a formula, say $H(n, v)$, with free variables n and v. When $n \in \mathbb{N}$ and u is a set we have

$$u = s_n \iff (n, u) \in w_n \iff \forall v\big(v{=}w_n \to (n, u) \in v\big) \iff \forall v\big(H(n, v) \to (n, u) \in v\big)$$

which can be expressed as a formula, say $G(n, u)$. Finally, to be careful, the statement $F(n, u)$ which is used in the Replacement Axiom, must have the property that for every set n (not necessarily with $n \in \mathbb{Z}$) there is a unique set u for which $F(n, u)$ is true, and so we take $F(n, u)$ to be the statement $(n{\in}\mathbb{N} \to G(n, u)) \wedge (\neg\, n{\in}\mathbb{N} \to u = \emptyset)$.

For our solution to be complete, we would need to prove that our statement $F(u, n)$ has the property that for every set n there is a unique set u such that $F(n, u)$ is true. To do this, we would first prove that for every $n \in \mathbb{N}$ there is a unique set v such that $H(n, v)$ is true. This can be proven using Induction.

3: In some books on set theory, the list of ZFC axioms includes an additional axiom called the Axiom of Regularity, which states that every nonempty set u contains an element v such that $u \cap v = \emptyset$. Assuming the Axiom of Regularity (along with the other ZFC axioms), prove each of the following statements.

(a) There does not exist a set u such that $u \in u$.

Solution: Suppose, for a contradiction, that u is a set with $u \in u$. Since $u \in \{u\}$ and $u \in u$ we have $u \in \{u\} \cap u$ and so $\{u\} \cap u \neq \emptyset$. Let $w = \{u\}$. By the Axiom of Regularity (applied to the set w) we can choose an element $v \in w$ such that $w \cap v = \emptyset$. Since $v \in w = \{u\}$ we must have $v = u$, so we have $\{u\} \cap u = w \cap v = \emptyset$. We have shown that $\{u\} \cap u = \emptyset$ and that $\{u\} \cap u \neq \emptyset$, so we have obtained the desired contradiction, hence there is no set u with $u \in u$.

(b) There do not exist sets u and v such that $u \in v$ and $v \in u$.

Solution: Suppose, for a contradiction, that u and v are sets with $u \in v$ and $v \in u$. Let $w = \{u, v\}$. By the Axiom of Regularity (applied to the set w), either $w \cap u = \emptyset$ or $w \cap v = \emptyset$. But since $u \in v$ and $u \in w$ we have $u \in w \cap v$ so $w \cap v \neq \emptyset$ and, similarly, since $v \in u$ and $v \in w$ we have $v \in w \cap u$ so that $w \cap u \neq \emptyset$. We have shown that either $w \cap u = \emptyset$ or $w \cap v = \emptyset$, and we have also shown that $w \cap u \neq \emptyset$ and $w \cap v \neq \emptyset$, so we have obtained the desired contradiction.

(c) For all sets u and v, if $u \cup \{u\} = v \cup \{v\}$ then $u = v$.

Solution: Let u and v be sets. Suppose that $u \cup \{u\} = v \cup \{v\}$. Suppose, for a contradiction, that $u \neq v$. Since $u \in u \cup \{u\}$ and $u \cup \{u\} = v \cup \{v\}$ we have $u \in v \cup \{v\}$. Since $u \in v \cup \{v\}$ it follows that either $u \in v$ or $u = v$. Since $u \neq v$ it follows that $u \in v$. A similar argument shows that $v \in u$. But then we have $u \in v$ and $v \in u$, which contradicts the result of Part (b).

(d) For all sets u, v, x and y, if $\big\{u, \{u, v\}\big\} = \big\{x, \{x, y\}\big\}$ then $u = x$ and $v = y$.

Solution: Let u, v, x, y be sets. Suppose that $\big\{u, \{u, v\}\big\} = \big\{x, \{x, y\}\big\}$. Note that $u \neq \{u, v\}$ because if we had $u = \{u, v\}$ then we would have $u \in \{u, v\} = u$ which contradicts Part (a). Similarly $x \neq \{x, y\}$ so the sets $\big\{u, \{x, y\}\big\}$ and $\big\{u, \{u, v\}\big\}$ are 2-element sets. Since $\big\{u, \{u, v\}\big\} = \big\{x, \{x, y\}\big\}$, with the sets on each side having 2 distinct elements, either $\big(u = x$ and $\{u, v\} = \{x, y\}\big)$ or $\big(u = \{x, y\}$ and $\{u, v\} = x\big)$.

Case 1: suppose that $u = x$ and $\{u, v\} = \{x, y\}$. We need to show that $v = y$. Since $v \in \{u, v\}$ and $\{u, v\} = \{x, y\}$ we have $v \in \{x, y\}$ hence either $v = x$ or $v = y$. If $v = y$ we are done, so suppose that $v = x$. Then we have $u = x = v$ hence $\{u, v\} = \{v\}$. Since $y \in \{x, y\} = \{u, v\} = \{v\}$ we have $y = v$, as required.

Case 2: suppose that $u = \{x, y\}$ and $\{u, v\} = x$. Since $u \in \{u, v\}$ and $\{u, v\} = x$ we have $u \in x$. Since $x \in \{x, y\}$ and $\{x, y\} = u$ we have $x \in u$. But then we have $u \in x$ and $x \in u$ which contradicts Part (b), and so Case 2 does not arise.