Chapter 5. Topology in Euclidean Space

Dot Product and Norm

5.1 Definition: For vectors $x, y \in \mathbb{R}^n$ we define the **dot product** of x and y to be

$$x \cdot y = y^T x = \sum_{i=1}^{n} x_i y_i \, .$$

5.2 Theorem: *(Properties of the Dot Product) For all $x, y, z \in \mathbb{R}^n$ and all $t \in \mathbb{R}$ we have*
(1) (Bilinearity) $(x + y) \cdot z = x \cdot z + y \cdot z \; , \; (tx) \cdot y = t(x \cdot y)$
$x \cdot (y + z) = x \cdot y + x \cdot z \; , \; x \cdot (ty) = t(x \cdot y),$
(2) (Symmetry) $x \cdot y = y \cdot x$, and
(3) (Positive Definiteness) $x \cdot x \geq 0$ with $x \cdot x = 0$ if and only if $x = 0$.

Proof: The proof is left as an exercise.

5.3 Definition: For a vector $x \in \mathbb{R}^n$, we define the **norm** (or **length**) of x to be

$$|x| = \sqrt{x \cdot x} = \sqrt{\textstyle\sum_{i=1}^{n} {x_i}^2}.$$

We say that x is a **unit vector** when $|x| = 1$.

5.4 Theorem: *(Properties of the Norm) Let $x, y \in \mathbb{R}^n$ and let $t \in \mathbb{R}$. Then*
(1) (Positive Definiteness) $|x| \geq 0$ with $|x| = 0$ if and only if $x = 0$,
(2) (Scaling) $|tx| = |t||x|$,
(3) $|x \pm y|^2 = |x|^2 \pm 2(x \cdot y) + |y|^2$.
(4) (The Polarization Identities) $x \cdot y = \frac{1}{2}\big(|x + y|^2 - |x|^2 - |y|^2\big) = \frac{1}{4}\big(|x + y|^2 - |x - y|^2\big)$,
(5) (The Cauchy-Schwarz Inequality) $|x \cdot y| \leq |x|\,|y|$ with $|x \cdot y| = |x|\,|y|$ if and only if the set $\{x, y\}$ is linearly dependent, and
(6) (The Triangle Inequality) $|x + y| \leq |x| + |y|$.

Proof: We leave the proofs of Parts (1), (2) and (3) as an exercise, and we note that (4) follows immediately from (3). To prove part (5), suppose first that $\{x, y\}$ is linearly dependent. Then one of x and y is a multiple of the other, say $y = tx$ with $t \in \mathbb{R}$. Then

$$|x \cdot y| = |x \cdot (tx)| = |t(x \cdot x)| = |t|\,|x|^2 = |x|\,|tx| = |x|\,|y|.$$

Suppose next that $\{x, y\}$ is linearly independent. Then for all $t \in \mathbb{R}$ we have $x + ty \neq 0$ and so

$$0 \neq |x + ty|^2 = (x + ty) \cdot (x + ty) = |x|^2 + 2t(x \cdot y) + t^2|y|^2.$$

Since the quadratic on the right is non-zero for all $t \in \mathbb{R}$, it follows that the discriminant of the quadratic must be negative, that is

$$4(x \cdot y)^2 - 4|x|^2|y|^2 < 0.$$

Thus $(x \cdot y)^2 < |x|^2|y|^2$ and hence $|x \cdot y| < |x|\,|y|$. This proves part (5).

Using part (5) note that

$$|x+y|^2 = |x|^2 + 2(x \cdot y) + |y|^2 \leq |x+y|^2 + 2|x \cdot y| + |y|^2 \leq |x|^2 + 2|x|\,|y| + |y|^2 = \big(|x| + |y|\big)^2$$

and so $|x + y| \leq |x| + |y|$, which proves part (6).

5.5 Definition: For points $a, b \in \mathbb{R}^n$, we define the **distance** between a and b to be

$$\text{dist}(a,b) = |b - a|.$$

5.6 Theorem: *(Properties of Distance) Let $a, b, c \in \mathbb{R}^n$. Then*
(1) (Positive Definiteness) $\text{dist}(a,b) \geq 0$ *with* $\text{dist}(a,b) = 0$ *if and only if* $a = b$,
(2) (Symmetry) $\text{dist}(a,b) = \text{dist}(b,a)$, *and*
(3) (The Triangle Inequality) $\text{dist}(a,c) \leq \text{dist}(a,b) + \text{dist}(b,c)$.

Proof: The proof is left as an exercise.

5.7 Definition: For nonzero vectors $0 \neq u, v \in \mathbb{R}^n$, we define the **angle between** u and v to be $\theta(u,v) = \cos^{-1} \frac{u \cdot v}{|u|\,|v|} \in [0, \pi]$. We say that u and v are **orthogonal** when $u \cdot v = 0$. As an exercise, determine (with proof) some properties of angles.

Open and Closed Sets

5.8 Definition: For $a \in \mathbb{R}^n$ and $0 < r \in \mathbb{R}$, the **sphere**, the **open ball**, the **closed ball**, and the (open) **punctured ball** in $\mathbb{R}^n$ centered at a of radius r are defined to be the sets

$$\begin{aligned}
S(a,r) &= \left\{x \in \mathbb{R}^n \middle| \text{dist}(x,a) = r\right\} = \left\{x \in \mathbb{R}^n \middle| |a - x| = r\right\}, \\
B(a,r) &= \left\{x \in \mathbb{R}^n \middle| \text{dist}(x,a) < r\right\} = \left\{x \in \mathbb{R}^n \middle| |a - x| < r\right\}, \\
\overline{B}(a,r) &= \left\{x \in \mathbb{R}^n \middle| \text{dist}(x,a) \leq r\right\} = \left\{x \in \mathbb{R}^n \middle| |a - x| \leq r\right\}, \\
B^*(a,r) &= \left\{x \in \mathbb{R}^n \middle| 0 < \text{dist}(x,a) < r\right\} = \left\{x \in \mathbb{R}^n \middle| 0 < |a - x| < r\right\}.
\end{aligned}$$

5.9 Definition: Let $A \subseteq \mathbb{R}^n$. We say that A is **bounded** when $A \subseteq B(a,r)$ for some $a \in \mathbb{R}^n$ and some $0 < r \in \mathbb{R}$. As an exercise, verify that A is bounded if and only if $A \subseteq B(0,r)$ for some $r > 0$.

5.10 Definition: For a set $A \subseteq \mathbb{R}^n$, we say that A is **open** (in $\mathbb{R}^n$) when for every $a \in A$ there exists $r > 0$ such that $B(a,r) \subseteq A$, and we say that A is **closed** (in $\mathbb{R}^n$) when its complement $A^c = \mathbb{R}^n \setminus A$ is open in $\mathbb{R}^n$.

5.11 Exercise: Show that in $\mathbb{R}$, open intervals are open, and closed intervals are closed.

5.12 Example: Show that for $a \in \mathbb{R}^n$ and $0 < r \in \mathbb{R}$, the set $B(a,r)$ is open and the set $\overline{B}(a,r)$ is closed.

Solution: Let $a \in \mathbb{R}^n$ and let $r > 0$. We claim that $B(a,r)$ is open. We need to show that for all $b \in B(a,r)$ there exists $s > 0$ such that $B(b,s) \subseteq B(a,r)$. Let $b \in B(a,r)$ and note that $|b - a| < r$. Let $s = r - |b - a|$ and note that $s > 0$. Let $x \in B(b,s)$, so we have $|x - b| < s$. Then, by the Triangle Inequality, we have

$$|x - a| = |x - b + b - a| \leq |x - b| + |b - a| < s + |b - a| = r$$

and so $x \in B(a,r)$. This shows that $B(b,s) \subseteq B(a,r)$ and hence $B(a,r)$ is open.

Next we claim that $\overline{B}(a,r)$ is closed, that is $\overline{B}(a,r)^c$ is open. Let $b \in \overline{B}(a,r)^c$, that is let $b \in \mathbb{R}^n$ with $b \notin \overline{B}(a,r)$. Since $b \notin \overline{B}(a,r)$ we have $|b - a| > r$. Let $s = |b - a| - r > 0$. Let $x \in B(b,s)$ and note that $|x - b| < s$. Then we have

$$|b - a| = |b - x + x - a| \leq |b - x| + |x - a| < s + |x - a|$$

and so $|x - a| > |b - a| - s = r$. Since $|x - a| > r$ we have $x \notin \overline{B}(a,r)$ and so $x \in \overline{B}(a,r)^c$. This shows that $B(b,s) \subseteq \overline{B}(a,r)^c$ and it follows that $\overline{B}(a,r)^c$ is open and hence that $\overline{B}(a,r)$ is closed.

5.13 Theorem: *(Basic Properties of Open Sets)*

(1) The sets $\emptyset$ and $\mathbb{R}^n$ are open in $\mathbb{R}^n$.
(2) If S is a set of open sets then the union $\bigcup S = \bigcup_{U\in S} U$ is open.
(3) If S is a finite set of open sets then the intersection $\bigcap S = \bigcap_{U\in S} U$ is open.

Proof: The empty set is open because any statement of the form "for all $x\in\emptyset$ F" (where F is any statement) is considered to be true (by convention). The set $\mathbb{R}^n$ is open because given $a \in \mathbb{R}^n$ we can choose any value of $r > 0$ and then we have $B(a,r) \subseteq \mathbb{R}^n$ by the definition of $B(a,r)$. This proves Part (1).

To prove Part (2), let S be any set of open sets. Let $a \in \bigcup S = \bigcup_{U\in S} U$. Choose an open set $U \in S$ such that $a \in U$. Since U is open we can choose $r > 0$ such that $B(a,r) \subseteq U$. Since $U \in S$ we have $U \subseteq \bigcup S$. Since $B(a,r) \subseteq U$ and $U \subseteq \bigcup S$ we have $B(a,r) \subseteq \bigcup S$. Thus $\bigcup S$ is open, as required.

To prove Part (3), let S be a finite set of open sets. If $S = \emptyset$ then we use the convention that $\bigcap S = \mathbb{R}^n$, which is open. Suppose that $S \neq \emptyset$, say $S = \{U_1, U_2, \cdots, U_m\}$ where each U_k is an open set. Let $a \in \bigcap S = \bigcap_{k=1}^m U_k$. For each index k, since $a \in U_k$ we can choose $r_k > 0$ so that $B(a, r_k) \subseteq U_k$. Let $r = \min\{r_1, r_2, \cdots, r_m\}$. Then for each index k we have $B(a,r) \subseteq B(a,r_k) \subseteq U_k$. Since $B(a,r) \subseteq U_k$ for every index k, it follows that $B(a,r) \subseteq \bigcap_{k=1}^m U_k = \bigcap S$. Thus $\bigcap S$ is open, as required.

5.14 Theorem: *(Basic Properties of Closed Sets)*

(1) The sets $\emptyset$ and $\mathbb{R}^n$ are closed in $\mathbb{R}^n$.
(2) If S is a set of closed sets then the intersection $\bigcap S = \bigcap_{K\in S} K$ is closed.
(3) If S is a finite set of closed sets then the union $\bigcup S = \bigcup_{K\in S} K$ is closed.

Proof: The proof is left as an exercise

Interior and Closure

5.15 Definition: Let $A \subseteq \mathbb{R}^n$. The **interior** and the **closure** of A (in $\mathbb{R}^n$) are the sets

$$A^0 = \bigcup \left\{ U \subseteq \mathbb{R}^n \middle| U \text{ is open, and } U \subseteq A \right\},$$
$$\overline{A} = \bigcap \left\{ K \subseteq \mathbb{R}^n \middle| K \text{ is closed and } A \subseteq K \right\}.$$

5.16 Theorem: *Let $A \subseteq \mathbb{R}^n$.*

(1) The interior of A is the largest open set which is contained in A. In other words, $A^0 \subseteq A$ and A^0 is open, and for every open set U with $U \subseteq A$ we have $U \subseteq A^0$.

(2) The closure of A is the smallest closed set which contains A. In other words, $A \subseteq \overline{A}$ and $\overline{A}$ is closed, and for every closed set K with $A \subseteq K$ we have $\overline{A} \subseteq K$.

Proof: Note that A^0 is open by Part (2) of Theorem 5.13, because A^0 is equal to the union of a set of open sets. Also note that $A^0 \subseteq A$ because A^0 is equal to the union of a set of subsets of A. Finally note that for any open set U with $U \subseteq A$ we have $U \in S$ so that $U \subseteq \bigcup S = A^0$. This completes the proof of Part (1), and the proof of Part (2) is similar.

5.17 Corollary: *Let $A \subseteq \mathbb{R}^n$.*
(1) $(A^0)^0 = A^0$ and $\overline{\overline{A}} = \overline{A}$.
(2) A is open if and only if $A = A^0$
(3) A is closed if and only if $A = \overline{A}$.

Proof: The proof is left as an exercise.

Interior Points, Limit Points and Boundary Points

5.18 Definition: Let $A\subseteq\mathbb{R}^n$. An **interior point** of A is a point $a\in A$ such that for some $r>0$ we have $B(a,r)\subseteq A$. A **limit point** of A is a point $a\in\mathbb{R}^n$ such that for every $r>0$ we have $B^*(a,r)\cap A\neq\emptyset$. An **isolated point** of A is a point $a\in A$ which is not a limit point of A. A **boundary point** of A is a point $a\in\mathbb{R}^n$ such that for every $r>0$ we have $B(a,r)\cap A\neq\emptyset$ and $B(a,r)\cap A^c\neq\emptyset$. The set of limit points of A is denoted by A'. The **boundary** of A, denoted by ∂A, is the set of all boundary points of A.

5.19 Theorem: *(Properties of Interior, Limit and Boundary Points) Let $A\subseteq\mathbb{R}^n$.*
(1) A^0 is equal to the set of all interior points of A.
(2) A is closed if and only if $A'\subseteq A$.
(3) $\overline{A}=A\cup A'$.
(4) $\partial A=\overline{A}\setminus A^0$.

Proof: We leave the proofs of Parts (1) and (4) as exercises. To prove Part (2) note that when $a\notin A$ we have $B(a,r)\cap A=B^*(a,r)\cap A$ and so

$$\begin{aligned}
A\text{ is closed} &\iff A^c\text{ is open}\\
&\iff \forall a\in A^c\ \exists r>0\ B(a,r)\subseteq A^c\\
&\iff \forall a\in\mathbb{R}^n\ \big(a\notin A\Longrightarrow \exists r>0\ B(a,r)\subseteq A^c\\
&\iff \forall a\in\mathbb{R}^n\ \big(a\notin A\Longrightarrow \exists r>0\ B(a,r)\cap A=\emptyset\big)\\
&\iff \forall a\in\mathbb{R}^n\ \big(a\notin A\Longrightarrow \exists r>0\ B^*(a,r)\cap A=\emptyset\big)\\
&\iff \forall a\in\mathbb{R}^n\ \big(\forall r>0\ B^*(a,r)\cap A\neq\emptyset\Longrightarrow a\in A\big)\\
&\iff \forall a\in\mathbb{R}^n\ \big(a\in A'\Longrightarrow a\in A\big)\\
&\iff A'\subseteq A.
\end{aligned}$$

To prove Part (3) we shall prove that $A\cup A'$ is the smallest closed set which contains A. It is clear that $A\cup A'$ contains A. We claim that $A\cup A'$ is closed, that is $(A\cup A')^c$ is open. Let $a\in(A\cup A')^c$, that is let $a\in\mathbb{R}^n$ with $a\notin A$ and $a\notin A'$. Since $a\notin A'$ we can choose $r>0$ so that $B(a,r)\cap A=\emptyset$. We claim that because $B(a,r)\cap A=\emptyset$ it follows that $B(a,r)\cap A'=\emptyset$. Suppose, for a contradiction, that $B(a,r)\cap A'\neq\emptyset$. Choose $b\in B(a,r)\cap A'$. Since $b\in B(a,r)$ and $B(a,r)$ is open, we can choose $s>0$ so that $B(b,s)\subseteq B(a,r)$. Since $b\in A'$ it follows that $B(b,s)\cap A\neq\emptyset$. Choose $x\in B(b,s)\cap A$. Then we have $x\in B(b,s)\subseteq B(a,r)$ and $x\in A$ and so $x\in B(a,r)\cap A$, which contradicts the fact that $B(a,r)\cap A=\emptyset$. Thus $B(a,r)\cap A'=\emptyset$, as claimed. Since $B(a,r)\cap A=\emptyset$ and $B(a,r)\cap A'=\emptyset$ it follows that $B(a,r)\cap(A\cup A')=\emptyset$ hence $B(a,r)\subseteq(A\cup A')^c$. Thus proves that $(A\cup A')^c$ is open, and hence $A\cup A'$ is closed.

It remains to show that for every closed set K with $A\subseteq K$ we have $A\cup A'\subseteq K$. Let K be a closed set in $\mathbb{R}^n$ with $A\subseteq K$. Note that since $A\subseteq K$ it follows that $A'\subseteq K'$ because if $a\in A'$ then for all $r>0$ we have $B(a,r)\cap A\neq\emptyset$ hence $B(a,r)\cap K\neq\emptyset$ and so $a\in K'$. Since K is closed we have $K'\subseteq K$ by Part (2). Since $A'\subseteq K'$ and $K'\subseteq K$ we have $A'\subseteq K$. Since $A\subseteq K$ and $A'\subseteq K$ we have $A\cup A'\subseteq K$, as required. This completes the proof of Part (3).

Connected Sets and Compact Sets

5.20 Definition: Let $A \subseteq \mathbb{R}^n$. For sets $U, V \subseteq \mathbb{R}^n$, we say that U and V **separate** A when

$$U \cap A \neq \emptyset \,,\, V \cap A \neq \emptyset \,,\, U \cap V = \emptyset \text{ and } A \subseteq U \cup V.$$

We say that A is **connected** when there do not exist open sets U and V in $\mathbb{R}^n$ which separate A. We say that A is **disconnected** when it is not connected, that is when there do exist open sets U and V in $\mathbb{R}^n$ which separate A.

5.21 Theorem: *The connected sets in $\mathbb{R}$ are the intervals, that is the sets of one of the forms $(a,b)\,,\,[a,b)\,,\,(a,b]\,,\,[a,b]\,,\,(a,\infty)\,,\,[a,\infty)\,,\,(-\infty,b)\,,\,(-\infty,b]\,,\,(-\infty,\infty)$ for some $a, b \in \mathbb{R}$ with $a \leq b$. We include the case that $a = b$ in order to include the degenerate intervals $\emptyset = (a,a)$ and $\{a\} = [a,a]$.*

Proof: We use the fact that the intervals in $\mathbb{R}$ are the sets with the intermediate value property (a set $A \subseteq \mathbb{R}$ has **the intermediate value property** when for all $a, b, \in A$ and all $x \in \mathbb{R}$, if $a < x < b$ then $x \in A$). Let $A \subseteq \mathbb{R}$. Suppose that A is not an interval. Then A does not have the intermediate value property so we can choose $a, b \in A$ and $u \in \mathbb{R}$ with $a < u < b$. Then $U = (-\infty, u)$ and $V = (u, \infty)$ separate A and so A is disconnected.

Suppose, conversely, that A is disconnected. Choose open sets U and V which separate A. Choose $a \in U$ and $b \in V$. Note that $a \neq b$ since $U \cap V = \emptyset$. Suppose that $a < b$ (the case that $b < a$ is similar). Let $u = \sup\big(U \cap [a,b]\big)$. Note that $u \neq a$ since we can choose $\delta > 0$ such that $[a, a+\delta) \subseteq U \cap [a,b]$ and then we have $u = \sup\big(U \cap [a,b]\big) \geq a + \delta$. Note that $u \neq b$ since we can choose $\delta > 0$ such that $(b-\delta, b] \subseteq V \cap [a,b]$ and then we have $u = \sup\big(U \cap [a,b]\big) \leq b - \delta$ since $U \cap V = \emptyset$. Thus we have $a < u < b$. Note that $u \notin U$ since if we had $u \in U$ we could choose $\delta > 0$ such that $(u-\delta, u+\delta) \subseteq U \cap [a,b]$ which contradicts the fact that $u = \sup\big(U \cap [a,b]\big)$. Note that $u \notin V$ since if we had $u \in V$ then we could choose $\delta > 0$ such that $(u-\delta, u+\delta) \subseteq V \cap [a,b]$ which contradicts the fact that $u = \sup\big(U \cap [a,b]\big)$ because $U \cap V = \emptyset$. Since $u \notin U$ and $u \notin V$ and $A \subseteq U \cap V$ we have $u \notin A$, so A does not have the intermediate value property, and so A is not an interval.

5.22 Definition: Let $A \subseteq \mathbb{R}^n$. An **open cover** of A is a set S of open sets in $\mathbb{R}^n$ such that $A \subseteq \bigcup S$. A **subcover** of an open cover S of A is a subset $T \subseteq S$ such that $A \subseteq \bigcup T$. We say that A is **compact** when every open cover of A has a finite subcover.

5.23 Exercise: Show that the set $A = \left\{\frac{1}{n}\middle| n \in \mathbb{Z}^+\right\}$ is not compact, but that the set $B = A \cup \{0\}$ is compact.

5.24 Theorem: *(The Nested Interval Theorem) Let $I_0, I_1, I_2, \cdots$ be nonempty, closed bounded intervals in $\mathbb{R}$. Suppose that $I_0 \supseteq I_1 \supset I_2 \supset \cdots$. Then $\bigcap_{k=0}^{\infty} I_k \neq \emptyset$.*

Proof: For each $k \geq 1$, let $I_k = [a_k, b_k]$ with $a_k < b_k$. For each k, since $I_{k+1} \subseteq I_k$ we have $a_k \leq a_{k+1} < b_{k+1} \leq b_{k+1}$. Since $a_k \geq a_{k+1}$ for all k, the sequence (a_k) is increasing. Since $a_k < b_k \leq b_{k-1} \leq \cdots \leq b_1$ for all k, the sequence (a_k) is bounded above by b_1. Since (a_k) is increasing and bounded above, it converges. Let $a = \sup\{a_k\} = \lim_{k\to\infty} a_k$. Similarly, (b_k) is decreasing and bounded below by a_1, and so it converges. Let $b = \inf\{b_k\} = \lim_{k\to\infty} b_k$. Fix $m \geq 1$. For all $k \geq m$ we have $a_m < b_m \leq b_{m+1} \leq \cdots \leq b_k$. Since $a_k \leq b_k$ for all k, by the Comparison Theorem we have $a \leq b$, and so the interval $[a,b]$ is not empty. Since (a_k) is increasing with $a_k \to a$, it follows (we leave the proof as an exercise) that $a_k \leq a$ for all $k \geq 1$. Similarly, we have $b_k \geq b$ for all $k \geq 1$ and so $[a,b] \subseteq [a_k, b_k] = I_k$. Thus $[a,b] \subseteq \bigcap_{k=1}^{\infty} I_k$, and so $\bigcap_{k=1}^{\infty} I_k \neq \emptyset$.

5.25 Definition: A **closed rectangle** in $\mathbb{R}^n$ is a set of the form

$$\begin{aligned} R &= [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \\ &= \big\{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \big| a_j \leq x_j \leq b_j \text{ for all } j\big\}. \end{aligned}$$

5.26 Theorem: *(Nested Rectangles) Let $R_1, R_2, R_3, \cdots$ be closed rectangles in $\mathbb{R}^n$ with $R_1 \supseteq R_2 \supseteq R_3 \supseteq \cdots$. Then*

$$\bigcap_{k=1}^{\infty} R_k \neq \emptyset.$$

Proof: Let $R_k = [a_{k,1}, b_{k,1}] \times [a_{k,2}, b_{k,2}] \times \cdots \times [a_{k,n}, b_{k,n}]$. Since $R_1 \supseteq R_2 \supseteq \cdots$ it follows that for each index j with $1 \leq j \leq n$ we have $[a_{1,j}, b_{1,j}] \supseteq [a_{2,j}, b_{2,j}] \supseteq \cdots$. By the Nested Interval Theorem, for each index j we can choose $u_j \in \bigcap\limits_{k=1}^{\infty} [a_{k,j}, b_{k,j}]$. Then for $u = (u_1, u_2, \cdots, u_n)$ we have $u \in \bigcap\limits_{k=1}^{\infty} R_k$.

5.27 Theorem: *(Compactness of Rectangles) Every closed rectangle in $\mathbb{R}^n$ is compact.*

Proof: Let $R = I_1 \times I_2 \times \cdots \times I_n$ where $I_j = [a_j, b_j]$ with $a_j \leq b_j$. Let d be the diameter of R, that is $d = \text{diam}(R) = \big(\sum\limits_{j=1}^{n} (b_j - a_j)^2\big)^{1/2}$. Let S be an open cover of R. Suppose, for a contradiction, that S does not have a finite subset which covers R. Let $a_{1,j} = a_j$, $b_{1,j} = b_j$, $I_{1,j} = I_j = [a_{1,j}, b_{1,j}]$ and $R_1 = R = I_{1,1} \times \cdots \times I_{1,n}$. Recursively, we construct rectangles $R = R_1 \supseteq R_2 \supseteq R_3 \supseteq \cdots$, with $R_k = I_{k,1} \times \cdots \times I_{k,n}$ where $I_{k,j} = [a_{k,j}, b_{k,j}]$, and $d_k = \text{diam}(R_k) = \big(\sum\limits_{j=1}^{n} (b_{k,j} - a_{k,j})^2\big)^{1/2} = \frac{d}{2^{k-1}}$, such that the open cover S does not have a finite subset which covers any of the rectangles R_k. We do this recursive construction as follows. Having constructed one of the rectangles R_k, we partition each of the intervals $I_{k,j} = [a_{k,j}, b_{k,j}]$ into the two equal-sized subintervals $[a_{k,j}, \frac{a_{k,j}+b_{k,j}}{2}]$ and $[\frac{a_{k,j}+b_{k,j}}{2}, b_{k,j}]$, and we thereby partition the rectangle R_k into 2^n equal-sized sub-rectangles. We choose R_{k+1} to be equal to one of these 2^n sub-rectangles with the property that the open cover S does not have a finite subset which covers R_{k+1} (if each of the 2^n sub-rectangles could be covered by a finite subset of S then the union of theses 2^n finite subsets would be a finite subset of S which covers R_k).

By the Nested Rectangles Theorem, we can choose an element $u \in \bigcap\limits_{k=1}^{\infty} R_k$. Since $u \in R$ and S covers R we can choose an open set $U \in S$ such that $u \in U$. Since U is open we can choose $r > 0$ such that $B(u, r) \subseteq U$. Since $d_k \to 0$ we can choose k so that $d_k < r$. Since $u \in R_k$ and $\text{diam} R_k = d_k < r$ we have $R_k \subseteq B(u, r) \subseteq U$. Thus S does have a finite subset, namely $\{U\}$, which covers R_k, giving the desired contradiction.

5.28 Theorem: *Let $A \subseteq K \subseteq \mathbb{R}^n$. If A is closed and K is compact then A is compact.*

Proof: Suppose that A is closed in $\mathbb{R}^n$ and that K is compact. Let S be an open cover of A. Let $A^c = \mathbb{R}^n \setminus A$. Since $A \subseteq \bigcup S$ we have $\bigcup S \cup \{A^c\} = \mathbb{R}^n$ and so $S \cup \{A^c\}$ is an open cover of K. Since K is compact, we can choose a finite subset $T \subseteq S \cup \{A^c\}$ with $K \subseteq \bigcup T$. Since $A \subseteq K \subseteq \bigcup T$ we also have $A \subseteq \bigcup \big(T \setminus \{A^c\}\big)$. Thus the open cover S of A does have a finite subcover, namely $T \setminus \{A^c\}$, and so A is compact, as required.

5.29 Theorem: *(The Heine-Borel Theorem) Let $A \subseteq \mathbb{R}^n$. Then A is compact if and only if A is closed and bounded.*

Proof: Suppose that A is compact. Suppose, for a contradiction, that A is not bounded. For each $k \in \mathbb{Z}^+$ let $U_k = B(0,k)$ and let $S = \{U_k \big| k \in \mathbb{Z}^+\}$. Then $\bigcup S = \mathbb{R}^n$ so S is an open cover of A. Let T be any finite subset of S. If $T = \emptyset$ then $\bigcup T = \emptyset$ and $A \not\subseteq \bigcup T$. Suppose that $T \neq \emptyset$, say $T = \{U_{k_1}, U_{k_2}, \cdots, U_{k_m}\}$ with $k_1 < k_2 < \cdots < k_m$. Since $U_{k_1} \subseteq U_{k_2} \subseteq \cdots \subseteq U_{k_m}$ we have $\bigcup T = \bigcup_{i=1}^m U_{k_i} = U_{k_m} = B(0, k_m)$. Since A is not bounded we have $A \not\subseteq B(0,k_m)$ and so $A \not\subseteq \bigcup T$. This shows that the open cover S has no finite subcover T, which contradicts the fact that A is compact.

Next suppose, for a contradiction, that A is not closed. By Part (2) of Theorem 5.19, it follows that $A' \not\subseteq A$. Choose $a \in A'$ with $a \notin A$. For each $k \in \mathbb{Z}^+$ let U_k be the open set $U_k = \overline{B}\big(a, \frac{1}{k}\big)^c = \{x \in \mathbb{R}^n \big| |x-a| > \frac{1}{k}\}$ and let $S = \{U_k \big| k \in \mathbb{Z}^+\}$. Note that $\bigcup S = \mathbb{R}^n \setminus \{a\}$ so S is an open cover of A. Let T be any finite subset of S. If $T = \emptyset$ then $\bigcup T = \emptyset$ so $A \not\subseteq \bigcup T$ (since A is not closed so $A \neq \emptyset$). Suppose that $T \neq \emptyset$, say $T = \{U_{k_1}, U_{k_2}, \cdots, U_{k_m}\}$ with $k_1 < k_2 < \cdots < k_m$. Since $U_{k_1} \subseteq U_{k_2} \subseteq \cdots \subseteq U_{k_m}$ we have $\bigcup T = \bigcup_{i=1}^m U_{k_i} = U_{k_m} = \overline{B}\big(a, \frac{1}{k_m}\big)^c$. Since $a \in A'$, we have $B^*\big(a, \frac{1}{k_m}\big) \cap A \neq \emptyset$ hence $\overline{B}\big(a, \frac{1}{k_m}\big) \cap A \neq \emptyset$ and so $A \not\subseteq \overline{B}\big(a, \frac{1}{k_m}\big)^c$, hence $A \not\subseteq \bigcup T$. This shows that the open cover S has no finite subcover T, which again contradicts the fact that A is compact.

Suppose, conversely, that A is closed and bounded. Since A is bounded we can choose $r > 0$ so that $A \subseteq B(0,r)$. Let R be the closed rectangle $R = \{x \in \mathbb{R}^n \big| |x_k| \leq r \text{ for all } k\}$. Note that $B(0,r) \subseteq R$ since when $x = (x_1, \cdots, x_n) \in B(0,r)$, for each index k we have

$$|x_k| = \big({x_k}^2\big)^{1/2} \leq \Big(\sum_{i=1}^n {x_i}^2\Big)^{1/2} = |x| < r.$$

Since A is closed and $A \subseteq R$ and R is compact, A is compact by the above theorem.

Topology in Subsets of Euclidean Space

5.30 Definition: Let $P \subseteq \mathbb{R}^n$. For $a \in P$ and $0 < r \in \mathbb{R}$ we define the **open ball in** P and the **closed ball in** P centred at a of radius r to be the sets

$$B_P(a,r) = \{x \in P \,\big|\, |x-a| < r\} = B(a,r) \cap P,$$
$$\overline{B}_P(a,r) = \{x \in P \,\big|\, |x-a| \leq r\} = \overline{B}(a,r) \cap P.$$

For $A \subseteq P \subseteq \mathbb{R}^n$, we say A is **open in** P when for every $a \in A$ there exists $r > 0$ such that $B_P(a,r) \subseteq A$, and we say A is **closed in** P when $A^c = P \setminus A$ is open in P.

5.31 Theorem: *Let $A \subseteq P \subseteq \mathbb{R}^n$.*

(1) A is open in P if and only if there exists an open set U in $\mathbb{R}^n$ such that $A = U \cap P$.
(2) A is closed in P if and only if there exists a closed set K in $\mathbb{R}^n$ such that $A = K \cap P$.

Proof: To prove Part (1), suppose first that A is open in P. For each $a \in A$, choose $r_a > 0$ so that $B(a, r_a) \cap P \subseteq A$, and let $U = \bigcup_{a \in A} B(a, r_a)$. Since U is equal to the union of a set of open sets in $\mathbb{R}^n$, it follows that U is open in $\mathbb{R}^n$. Note that $A \subseteq U \cap P$ and, since $B(a, r_a) \cap P \subseteq A$ for every $a \in A$, we also have $U \cap P = \big(\bigcup_{a \in U} B(a, r_a)\big) \cap P = \bigcup_{a \in A} \big(B(a, r_a) \cap P\big) \subseteq A$. Thus $A = U \cap P$, as required.

Suppose, conversely, that $A = U \cap P$ with U open in $\mathbb{R}^n$. Let $a \in A$. Since $a \in A = U \cap P$, we also have $a \in U$. Since $a \in U$ and U is open in $\mathbb{R}^n$ we can choose $r > 0$ so that $B(a,r) \subseteq U$. Since $B(a,r) \subseteq U$ and $U \cap P = A$ we have $B(a,r) \cap P \subseteq U \cap P = A$, as required.

To prove Part (2), suppose first that A is closed in P. Let B be the complement of A in P, that is $B = P \setminus A$. Then B is open in P. Choose an open set U in $\mathbb{R}^n$ such that $B = U \cap P$. Let K be the complement of U in $\mathbb{R}^n$, that is $K = \mathbb{R}^n \setminus U$. Then $A = K \cap P$ since for $x \in \mathbb{R}^n$ we have $x \in A \iff \big(x \in P \text{ and } x \notin B\big) \iff \big(x \in P \text{ and } x \notin U \cap P\big) \iff \big(x \in P \text{ and } x \notin U\big) \iff \big(x \in P \text{ and } x \in K\big) \iff x \in K \cap P$.

Suppose, conversely, that K is a closed set in P with $A = K \cap P$. Let B be the complement of A in P, that is $B = P \setminus A$, and let U be the complement of K in P, that is $U = P \setminus K$, and note that U is open in P. Then we have $B = U \cap P$ since for $x \in P$ we have $x \in B \iff \big(x \in P \text{ and } x \notin A\big) \iff \big(x \in P \text{ and } x \notin K \cap P\big) \iff \big(x \in P \text{ and } x \notin K\big) \iff \big(x \in P \text{ and } x \in U\big) \iff x \in U \cap P$. Since U is open in P and $B = U \cap P$ we know that B is open in P. Since B is open in P, its complement $A = P \setminus B$ is closed in P.

5.32 Theorem: *Let $A \subseteq P \subseteq \mathbb{R}^n$. Define A to be* **connected in** *P when there do not exists sets $E, F \subseteq P$ which are open in P and which separate A. Define A to be* **compact in** *P when for every set S of open sets in P such that $A \subseteq \bigcup S$ there exists a finite subset $T \subseteq S$ such that $A \subseteq \bigcup T$. Then*

(1) A is connected in P if and only if A is connected in $\mathbb{R}^n$, and
(2) A is compact in P if and only if A is compact in $\mathbb{R}^n$.

Proof: We prove. Part (1) and leave the proof of Part (2) as an exercise. Suppose that A is not connected in $\mathbb{R}^n$. Choose open sets U and V in $\mathbb{R}^n$ which separate A, that is $U \cap A \neq \emptyset$, $V \cap A \neq \emptyset$, $U \cap V = \emptyset$ and $A \subseteq U \cup V$. Let $E = U \cap P$ and $F = V \cap P$. Note that E and F are open in P and E and F separate A.

Suppose, conversely, that there exist sets $E, F \subseteq P$ which are open in P and which separate A, that is $A \cap E \neq \emptyset$, $A \cap F \neq \emptyset$, $E \cap F = \emptyset$ and $A \subseteq E \cup F$. Choose open sets $U, V \subseteq \mathbb{R}^n$ such that $E = U \cap P$ and $F = V \cap P$. Note that it is possible that $U \cap V \neq \emptyset$ and so U and V might not separate A in $\mathbb{R}^n$. For this reason, we shall construct open subsets $U_0 \subseteq U$ and $V_0 \subseteq V$ which do separate A in $\mathbb{R}^n$. For each $a \in E$ choose $r_a > 0$ such that $B(a, 2r_a) \subseteq U$ and then let $U_0 = \bigcup_{a \in E} B(a, r_a)$. Note that U_0 is open in $\mathbb{R}^n$ (since it is a union of open sets in $\mathbb{R}^n$) and that we have $E \subseteq U_0 \subseteq U$. Similarly, for each $b \in F$ choose $s_b > 0$ so that $B(b, 2s_b) \subseteq V$, and then let $V_0 = \bigcup_{b \in F} B(b, s_b)$. Note that V_0 is open in $\mathbb{R}^n$ and $F \subseteq V_0 \subseteq V$. We claim that the open sets U_0 and V_0 separate A in $\mathbb{R}^n$. Since $E \subseteq U_0$ and $F \subseteq V_0$ we have $\emptyset \neq A \cap E \subseteq A \cap U_0$, $\emptyset \neq A \cap F \subseteq A \cap V_0$ and $A \subseteq E \cup F \subseteq U_0 \cup V_0$. It remains to show that $U_0 \cap V_0 = \emptyset$. Suppose, for a contradiction, that $U_0 \cap V_0 \neq \emptyset$. Choose $x \in U_0 \cap V_0$. Since $x \in U_0 = \bigcup_{a \in E} B(a, r_a)$ we can choose $a \in E$ such that $x \in B(a, r_a)$. Similarly, we can choose $b \in F$ so that $x \in B(b, s_b)$. Suppose that $r_a \geq s_b$ (the case that $s_b \geq r_a$ is similar). By the Triangle Inequality, it follows that $|b - a| \leq |b - x| + |x - a| < s_b + r_a \leq 2r_a$ and so we have $b \in B(a, 2r_a) \subseteq U$. Since $b \in F \subseteq P$ and $b \in U$ we have $b \in U \cap P = E$. Thus we have $b \in E \cap F$ which contradicts the fact that $E \cap F = \emptyset$, and so $U_0 \cap V_0 = \emptyset$, as required.

5.33 Corollary: *A set $A \subseteq \mathbb{R}^n$ is connected (in $\mathbb{R}^n$) if and only if the only subsets of A which are both open and closed in A are the sets $\emptyset$ and A.*

Proof: We leave it as an exercise to show that this follows from the above theorem by taking $A = P$.

Limits of Sequences

5.34 Definition: For $p \in \mathbb{Z}$, let $\mathbb{Z}_{\geq p} = \big\{n \in \mathbb{Z} | n \geq p\big\} = \{p, p+1, p+2, \cdots\}$. For a set A, a **sequence** in A is a function $x : \mathbb{Z}_{\geq p} \to A$ for some $p \in \mathbb{Z}$. We write $(x_n)_{n\geq p}$ to denote the sequence $x : \mathbb{Z}_{\geq p} \to A$ given by $x(n) = x_n$, where $x_n \in A$ for all $n \geq p$. A **subsequence** of the sequence $(x_n)_{n\geq p}$ is a sequence of the form $(y_k)_{k\geq q}$ with $y_k = x_{n_k}$ for some $p \leq n_k < n_{k+1}$ for all $k \geq q$.

5.35 Definition: Let $(x_n)_{n\geq p}$ be a sequence in $\mathbb{R}^m$. We say the sequence $(x_n)_{n\geq p}$ is **bounded** when

$$\exists r > 0\ \forall n \in \mathbb{Z}_{\geq p}\ |a_n| \leq r.$$

For $a \in \mathbb{R}^m$, we say that the sequence $(x_n)_{n\geq p}$ **converges to** a and write $\lim\limits_{n\to\infty} x_n = a$ (or $x_n \to a$) when

$$\forall \epsilon > 0\ \exists N \in \mathbb{Z}_{\geq p}\ \forall n \in \mathbb{Z}_{\geq p}\big(n \geq N \Longrightarrow |x_n - a| < \epsilon\big).$$

We say the sequence $(x_n)_{n\geq p}$ **diverges to** ∞ and write $\lim\limits_{n\to\infty} x_n = \infty$ (or $x_n \to \infty$) when

$$\forall r > 0\ \exists N \in \mathbb{Z}_{\geq p}\ \forall n \in Z_{\geq p}\big(n \geq N \Longrightarrow |x_n| \geq r\big).$$

We say that the sequence $(x_n)_{n\geq p}$ **converges** when it converges to some point $a \in \mathbb{R}^m$ and otherwise we say that it **diverges**.

5.36 Theorem: *(Convergent Sequences are Bounded) Let $(x_n)_{n\geq p}$ be a sequence in $\mathbb{R}^m$. If $(x_n)_{n\geq p}$ converges in $\mathbb{R}^m$ then $(x_n)_{n\geq p}$ is bounded.*

Proof: Suppose that $(x_n)_{n\geq p}$ converges in $\mathbb{R}^m$. Let $a = \lim\limits_{n\to\infty} x_n \in \mathbb{R}^m$. Choose $N \geq p$ such that $n \geq N \Longrightarrow |x_n - a| < 1$. For $n \geq N$, by the Triangle Inequality we have $|x_n| \leq |x_n - a| + |a| < 1 + |a|$. Thus we can choose $r = \max\big\{|x_p|, |x_{p+1}|, \cdots, |x_{N-1}|\,,\ 1+|a|\big\}$ to obtain $|x_n| \leq r$ for all $n \geq p$, and so the sequence $(x_n)_{n\geq p}$ is bounded, as required.

5.37 Theorem: *(Uniqueness of Limits of Sequences) Let $(x_n)_{n\geq p}$ be a sequence in $\mathbb{R}^m$ and let $a, b \in \mathbb{R}^m \cup \{\infty\}$. If $\lim\limits_{n\to\infty} x_n = a$ and $\lim\limits_{n\to\infty} x_n = b$ then $a = b$.*

Proof: We prove the theorem in the case that $a, b \in \mathbb{R}^m$ and leave the case that $a = \infty$ or $b = \infty$ as an exercise. Suppose that $\lim\limits_{n\to\infty} x_n = a \in \mathbb{R}^m$ and $\lim\limits_{n\to\infty} x_n = b \in \mathbb{R}^m$. Suppose, for a contradiction, that $a \neq b$. Choose $N_1 \geq p$ such that $n \geq N_1 \Longrightarrow |x_n - a| < \frac{|a-b|}{2}$ and choose $N_2 \geq p$ such that $n \geq N_2 \Longrightarrow |x_n - b| < \frac{|a-b|}{2}$. Let $N = \max\{N_1, N_2\}$. For $n \geq N$ we have $|a - b| \leq |a - x_n| + |x_n - b| < \frac{|a-b|}{2} + \frac{|a-b|}{2} = |a - b|$ which is impossible. Thus we must have $a = b$, as required.

5.38 Theorem: *(Limits of Subsequences) Let $(x_n)_{n\geq p}$ be a sequence in $\mathbb{R}^m$ and let $(x_{n_k})_{k\geq q}$ be a subsequence of $(x_n)_{n\geq p}$. If $\lim\limits_{n\to\infty} x_n = a \in \mathbb{R}^m \cup \{\infty\}$ then $\lim\limits_{k\to\infty} x_{n_k} = a$.*

Proof: We give the proof in the case that $a \in \mathbb{R}^m$. Suppose that $\lim\limits_{n\to\infty} x_n = a \in \mathbb{R}^m$ and let $(x_{n_k})_{k\geq q}$ be any subsequence of (x_n). Let $\epsilon > 0$. Choose $N \geq p$ such that $n \geq N \Longrightarrow |x_n - a| < \epsilon$. Choose $M \geq q$ such that $k \geq M \Longrightarrow n_k \geq N$ (we can do this since each $n_k \in \mathbb{Z}$ with $n_k < n_{k+1}$ and hence $n_k \to \infty$ as $k \to \infty$). Then for $k \geq M$ we have $n_k \geq N$ and so $|x_{n_k} - a| < \epsilon$. Thus $\lim\limits_{k\to\infty} x_{n_k} = a$, as required.

5.39 Remark: It follows from the above theorem that the initial index p of a sequence $(x_n)_{n\geq p}$ does not affect whether or not the sequence converges, and it does not influence the value of the limit. For this reason, we often omit the initial index p from our notation and denote the sequence $(x_n)_{n\geq p}$ simply as (x_n).

5.40 Definition: Let $(x_n)_{n\geq p}$ be a sequence in $\mathbb{R}^m$. For $n\geq p$ let $x_n=(x_{n,1},x_{n,2},\cdots,x_{n,m})$. For each index k with $1\leq k\leq m$, the k^{th} **component sequence** of $(x_n)_{n\geq p}$ is the sequence $(x_{n,k})_{n\geq p}=(x_{p,k},x_{p+1,k}\cdots)$. Note that the sequence $(x_n)_{n\geq p}$ in $\mathbb{R}^m$ determines and is determined by its component sequences $(x_{n,k})_{n\geq p}$.

5.41 Theorem: *(Limits of Component Sequences) Let $(x_n)_{n\geq p}$ be a sequence in $\mathbb{R}^m$, say $x_n=\big(x_{n,1},x_{n,2},\cdots,x_{n,m}\big)\in\mathbb{R}^m$.*
(1) $(x_n)_{n\geq p}$ is bounded if and only if $(x_{n,k})_{n\geq p}$ is bounded for all indices k.
(2) For $a=(a_1,\cdots,a_m)\in\mathbb{R}^m$ we have $\lim\limits_{n\to\infty}x_n=a$ if and only if $\lim\limits_{n\to\infty}x_{n,k}=a_k$ for all k.

Proof: Suppose that $(x_n)_{n\geq p}$ is bounded. Choose $r>0$ such that $|x_n|\leq r$ for all $n\geq p$. Let $n\geq p$ and let $1\leq k\leq m$. Then $|x_{n,k}|\leq|x_n|\leq r$ and so the sequence $(x_{n,k})_{n\geq p}$ is also bounded. Now suppose, conversely, that $(x_{n,k})_{n\geq p}$ is bounded for all indices k. For each k, chose $r_k>0$ such that $|x_{n,k}|\leq r_k$ for all $n\geq p$. Let $r=r_1+\cdots+r_m$. Then for all $n\geq p$, by the Triangle Inequality we have $|x_n|\leq|x_{n,1}|+|x_{n,2}|+\cdots+|x_{n,m}|<r_1+r_2+\cdots+r_m=r$ and so the sequence $(x_n)_{n\geq p}$ is bounded. This proves Part (1).

To prove Part (2), suppose first that $\lim\limits_{n\to\infty}x_n=a$. Let $\epsilon>0$ and choose $N\geq p$ so that $n\geq N\Longrightarrow\big|x_n-a\big|<\epsilon$. Let $1\leq k\leq m$. For $n\geq N$ we have $\big|x_{n,k}-a_k\big|\leq|x_n-a|<\epsilon$ and so $\lim\limits_{n\to\infty}x_{n,k}=a_k$. Now suppose, conversely, that $\lim\limits_{n\to\infty}x_{n,k}=a_k$ for all indices k. Let $\epsilon>0$. For each index k, choose $N_k\geq p$ such that $n\geq N_k\Longrightarrow\big|x_{n,k}-a_k\big|<\frac{\epsilon}{m}$. Then for $n\geq N$, by the Triangle Inequality we have $|x_n-a|\leq\sum\limits_{k=1}^{m}|x_{n,k}-a_k|<\epsilon$ and so $\lim\limits_{n\to\infty}x_n=a$.

5.42 Theorem: *(Operations on Limits of Sequences) Let (x_n) and (y_n) be sequences in $\mathbb{R}^m$ and let $c\in\mathbb{R}$. Suppose that $\lim\limits_{n\to\infty}x_n=u\in\mathbb{R}^m$ and $\lim\limits_{n\to\infty}y_n=v\in\mathbb{R}^m$. Then*

(1) $\lim\limits_{n\to\infty}(x_n+y_n)=u+v$,
(2) $\lim\limits_{n\to\infty}(c\,x_n)=c\,u$,
(3) $\lim\limits_{n\to\infty}|x_n|=|u|$,
(4) $\lim\limits_{n\to\infty}(x_n\cdot y_n)=u\cdot v$, and
(5) if $m=3$ then $\lim\limits_{n\to\infty}(x_n\times y_n)=u\times v$.

Proof: These follow easily from Part (2) of the above theorem and from known properties of sequences in $\mathbb{R}$. For example, to prove Part (1), note that

$$\lim_{n\to\infty}(x_n+y_n)_k=\lim_{n\to\infty}(x_{n,k}+y_{n,k})=\lim_{n\to\infty}x_{n,k}+\lim_{n\to\infty}y_{n,k}=u_k+v_k=(u+v)_k.$$

5.43 Theorem: *(Sequential Characterization of Limit Points) Let $A \subseteq \mathbb{R}^m$ and let $a \in \mathbb{R}^m$. Then $a \in A'$ if and only if there exists a sequence (x_n) in $A \setminus \{a\}$ such that $\lim\limits_{n\to\infty} x_n = a$.*

Proof: Let $a \in A'$. For each $n \in \mathbb{Z}^+$, since $a \in A'$ we have $B^*\big(a, \frac{1}{n}\big) \cap A \neq \emptyset$ so we can choose an element $x_n \in B^*\big(a, \frac{1}{n}\big) \cap A$ and then we have $x_n \in A \setminus \{a\}$ and $|x_n - a| < \frac{1}{n}$. Given $\epsilon > 0$ we can choose a positive integer $N > \frac{1}{\epsilon}$ and then we have $n \geq N \Longrightarrow |x_n - a| < \frac{1}{n} \leq \frac{1}{N} < \epsilon$. Thus $(x_n)_{n\geq 1}$ is a sequence in $A \setminus \{a\}$ with $\lim\limits_{n\to\infty} x_n = a$.

Suppose, conversely, that $(x_n)_{n\geq p}$ is a sequence in $A \setminus \{a\}$ with $\lim\limits_{n\to\infty} x_n = a$. Let $r > 0$. Since $\lim\limits_{n\to\infty} x_n = a$ we can choose $N \geq p$ so that $n \geq N \Longrightarrow |x_n - a| < r$. Then we have $x_N \in A \setminus A$ and $|x_N - a| < r$ so that $x_N \in B^*\big(a, r\big)$, and hence $B^*\big(a, r\big) \neq 0$. Since $r > 0$ was arbitrary, it follows that $a \in A'$.

5.44 Theorem: *(Sequential Characterization of Closed Sets) Let $A \subseteq \mathbb{R}^m$. Then A is closed (in $\mathbb{R}^m$) if and only if every for every sequence in A which converges in $\mathbb{R}^m$, the limit of the sequence lies in A.*

Proof: Suppose that A is closed. Let $(x_n)_{n\geq p}$ be a sequence in A which converges in $\mathbb{R}^n$. Let $a = \lim\limits_{n\to\infty} x_n$. Suppose, for a contradiction, that $a \notin A$. Since $a \notin A$ we have $A = A \setminus \{a\}$ and so (x_n) is a sequence in $A \setminus \{a\}$. Since (x_n) is a sequence in $A \setminus \{a\}$ with $\lim\limits_{n\to\infty} x_n = a$, we have $a \in A'$ by the Characterization of Limit Points. Since A is closed we have $A' \subseteq A$ and so $a \in A$, giving the desired contradiction.

Suppose, conversely, that for every sequence in A which converges in $\mathbb{R}^n$, the limit of the sequence lies in A. Let $a \in A'$. By the Characterization of Limit Points, we can choose a sequence (x_n) in $A \setminus \{a\}$ such that $\lim\limits_{n\to\infty} x_n = a$. Then (x_n) is a sequence in A which converges in $\mathbb{R}^n$, and so its limit must lie in A, thus we have $a \in A$. Since $a \in A'$ was arbitrary, this proves that $A' \subseteq A$ and so A is closed.

5.45 Theorem: *(Bolzano-Weierstrass) Every bounded sequence in $\mathbb{R}^m$ has a convergent subsequence.*

Proof: For this proof, we shall label the components of an element in $\mathbb{R}^m$ using superscripts rather than subscripts, so we shall write an element $x \in \mathbb{R}^m$ as $(x^1, x^2, \cdots, x^m)$. Let (x_n) be a bounded sequence in $\mathbb{R}^m$. Then the first component sequence (x_n^1) is a bounded sequence in $\mathbb{R}$. By the Bolzano-Weierstrass Theorem for sequences in $\mathbb{R}$, we can choose a convergent subsequence $(x_{n_\ell}^1)$, where $n_1 < n_2 < \cdots$. Since the second component sequence (x_n^2) is bounded, the subsequence $(x_{n_\ell}^2)$ is also bounded so we can choose a convergent subsequence $(x_{n_{\ell_k}}^2)$, where $\ell_1 < \ell_2 < \cdots$. Note that the sequence $(x_{n_{\ell_k}}^1)$ also converges because it is a subsequence of the convergent subsequence $(x_{n_\ell}^1)$. Since the sequence (x_n^3) is bounded, the subsequence $(x_{n_{\ell_k}}^3)$ is also bounded so we can choose a convergent subsequence $(x_{n_{\ell_{k_j}}}^3)$, where $k_1 < k_2 < \cdots$. We then obtain convergent subsequences of each of the first 3 component sequences (x_n^i) for $i = 1, 2, 3$, namely the subsequences $(x_{n_{\ell_{k_j}}}^i)$. We repeat the procedure until we obtain simultaneous subsequences of all m component sequences (x_n^i), which we can combine to form a subsequence of the original sequence (x_n) in $\mathbb{R}^m$.

5.46 Definition: Let $(x_n)_{n\geq p}$ be a sequence in $\mathbb{R}^m$. We say that (x_n) is **Cauchy** when

$$\forall\,\epsilon>0\ \exists\,N\in\mathbb{Z}_{\geq p}\ \forall\,k,\ell\in\mathbb{Z}_{\geq p}\ \Big(k,\ell\geq N\Longrightarrow |x_k-x_\ell|<\epsilon\Big).$$

5.47 Theorem: *(The Completeness of $\mathbb{R}^m$) For every sequence in $\mathbb{R}^m$, the sequence converges if and only if it is Cauchy.*

Proof: Let (x_n) be a sequence in $\mathbb{R}^m$. Suppose that (x_n) converges. Let $a=\lim\limits_{n\to\infty}x_n$. Let $\epsilon>0$. Choose N so that $n\geq N\Longrightarrow |x_n-a|<\frac{\epsilon}{2}$. Then for $k,\ell\geq N$ we have $|x_k-a|<\frac{\epsilon}{2}$ and $|x_\ell-a|<\frac{\epsilon}{2}$ so $|x_k-x_\ell|\leq|x_k-a|+|a-x_\ell|<\epsilon$. Thus (x_n) is Cauchy.

Now suppose that $(x_n)_{n\geq p}$ is Cauchy. Choose $N\geq p$ so that $k,\ell\geq N\Longrightarrow|x_k-x_\ell|<1$. Then for all $k\geq N$ we have $|x_k-x_N|<1$ hence $|x_k|\leq|x_k-x_N|+|x_N|<1+|x_N|$, and so (x_n) is bounded by $\max\big\{|x_p|,|x_{p+1}|,\cdots,|x_{N-1}|\,,\,1+|x_N|\big\}$. Choose a convergent subsequence (x_{n_k}) and let $a=\lim\limits_{k\to\infty}x_{n_k}$. Let $\epsilon>0$. Since (x_n) is Cauchy we can choose M so that $n,\ell\geq M\Longrightarrow|x_n-x_\ell|<\frac{\epsilon}{2}$. Since $\lim\limits_{k\to\infty}x_{n_k}=a$ we can choose k so that $n_k\geq M$ and $|x_{n_k}-a|<\frac{\epsilon}{2}$. Then for $n\geq M$ we have $|x_n-a|\leq|x_n-x_{n_k}|+|x_{n_k}-a|<\epsilon$.

Limits of Functions

5.48 Definition: Let $A\subseteq\mathbb{R}^\ell$ and let $f:A\to\mathbb{R}^m$. When a is a limit point of A and $b\in\mathbb{R}^m$, we say that $f(x)$ **converges to** b as x tends to a, and we write $\lim\limits_{x\to a}f(x)=b$ when

$$\forall\,\epsilon>0\ \exists\,\delta>0\ \forall\,x\in A\ \Big(0<|x-a|<\delta\Longrightarrow|f(x)-b|<\epsilon\Big).$$

When a is a limit point of A, we say that $f(x)$ **diverges to** ∞ and we write $\lim\limits_{x\to a}f(x)=\infty$ when

$$\forall\,r>0\ \exists\,\delta>0\ \forall\,x\in A\ \Big(0<|x-a|<\delta\Longrightarrow|f(x)|\geq r\Big).$$

5.49 Theorem: *(Sequential Characterization of Limits) Let $f:A\subseteq\mathbb{R}^\ell\to\mathbb{R}^m$, let a be a limit point of A and let $u\in\mathbb{R}^m\cup\{\infty\}$. Then $\lim\limits_{x\to a}f(x)=u$ if and only if $\lim\limits_{n\to\infty}f(x_n)=u$ for every sequence (x_n) in $A\setminus\{a\}$ with $\lim\limits_{n\to\infty}x_n=a$.*

Proof: We give the proof in the case that $u\in\mathbb{R}^m$. Suppose first that $\lim\limits_{x\to a}f(x)=u\in\mathbb{R}^m$. Let (x_n) be a sequence in $A\setminus\{a\}$ with $x_n\to a$. Let $\epsilon>0$. Since $\lim\limits_{x\to a}f(x)=u$ we can choose $\delta>0$ so that $0<|x-a|<\delta\Longrightarrow|f(x)-u|<\epsilon$. Since $x_n\to a$ we can choose N so that $n\geq N\Longrightarrow|x_n-a|<\delta$. For $n\geq N$ we have $|x_n-a|<\delta$ and we have $x_n\neq a$ (since $x_n\in A\setminus\{a\}$) and so $0<|x_n-a|<\delta$ and hence $|f(x_n)-u|<\epsilon$. Thus $\lim\limits_{n\to\infty}f(x_n)=u$, as required.

Suppose, conversely, that $\lim\limits_{x\to a}f(x)\neq u$. Choose ϵ such that for every $\delta>0$ there exists $x\in A$ such that $0<|x-a|<\delta$ and $|f(x)-u|\geq\epsilon$. For each $n\in\mathbb{Z}^+$, choose $x_n\in A$ such that $0<|x_n-a|<\frac{1}{n}$ and $|f(x_n)-u|\geq\epsilon$. For each n, since $0<|x_n-a|$ we have $x_n\neq a$ so the sequence (x_n) lies in $A\setminus\{a\}$. Since $|x_n-a|<\frac{1}{n}$ for all $n\in\mathbb{Z}^+$ it follows that $x_n\to a$. Since $|f(x_n)-u|\geq\epsilon$ for all n, it follows that $\lim\limits_{n\to\infty}f(x_n)\neq u$. Thus we have found a sequence (x_n) in $A\setminus\{a\}$ with $x_n\to a$ such that $\lim\limits_{n\to\infty}f(x_n)\neq u$.

5.50 Note: Using the Sequential Characterization of Limits, many properties of limits of sequences immediately imply analogous properties of limits of function. We list some of these properties in the following theorems.

5.51 Theorem: *(Uniqueness of Limits of Functions) Let $f : A \subseteq \mathbb{R}^\ell \to \mathbb{R}^m$, let $a \in A'$, and let $u, v \in \mathbb{R}^m \cup \{\infty\}$. If $\lim\limits_{x \to a} f(x) = u$ and $\lim\limits_{x \to a} f(x) = v$ then $u = v$.*

Proof: This can be proven by imitating the proof of the Uniqueness of Limits of Sequences. Alternatively, we can use Uniqueness of Limits of Sequences together with the Sequential Characterization of Limits as follows. Since $a \in A'$ we can choose a sequence $(x_n) \in A \setminus \{a\}$ such that $x_n \to a$. By the Sequential Characterization of Limits, since $\lim\limits_{x \to a} f(x) = u$ we have $\lim\limits_{n \to \infty} f(x_n) = u$ and since $\lim\limits_{x \to a} f(x) = v$ we have $\lim\limits_{n \to \infty} f(x_n) = v$. By the Uniqueness of Limits of Sequences, since $\lim\limits_{n \to \infty} f(x_n) = u$ and $\lim\limits_{n \to \infty} f(x_n) = v$ it follows that $u = v$.

5.52 Theorem: *(Local Determination of Limits of Functions) Let $A \subseteq \mathbb{R}^\ell$, let $a \in A'$, let $B = B^*(a, r) \cap A$ with $r > 0$. Let $f : A \to \mathbb{R}^m$ and let $g : B \to \mathbb{R}^m$ and suppose that $f(x) = g(x)$ for all $x \in B$. Then $\lim\limits_{x \to a} f(x)$ exists in $\mathbb{R}^m \cup \{\infty\}$ if an only if $\lim\limits_{x \to a} g(x)$ exists in $\mathbb{R}^m \cup \{\infty\}$ and, in this case, the limits are equal.*

Proof: We leave the proof as an exercise.

5.53 Definition: Let $f : A \subseteq \mathbb{R}^\ell \to \mathbb{R}^m$. We can write $f(x) = \big(f_1(x), f_2(x), \cdots, f_m(x)\big)$ where $f_k : A \to \mathbb{R}$ for each index k. Then the function f_k is called the k^{th} component function of f. Note that $f_k = p_k \circ f$ where $p_k : \mathbb{R}^m \to \mathbb{R}$ is the k **projection map** given by $p_k(y_1, \cdots, y_k, \cdots, y_m) = y_k$.

5.54 Theorem: *(Limits of Component Functions) Let $f : A \subseteq \mathbb{R}^\ell \to \mathbb{R}^m$ be given by $f(x) = \big(f_1(x), \cdots, f_m(x)\big)$, let a be a limit point of A, and let $b = (b_1, b_2, \cdots, b_m) \in \mathbb{R}^m$. Then $\lim\limits_{x \to a} f(x) = b$ if and only if $\lim\limits_{x \to a} f_k(x) = b_k$ for all indices k.*

Proof: Suppose that $\lim\limits_{x \to a} f(x) = b$. Let (x_n) be any sequence in $A \setminus \{a\}$ with $x_n \to a$. By the Sequential Characterization of Limits, we have $\lim\limits_{n \to \infty} f(x_n) = b$. By Limits of Component Sequences, we have $\lim\limits_{n \to \infty} f_k(x_n) = b_k$ for all indices k. By the Sequential Characterization of Limits again, it follows that $\lim\limits_{x \to a} f_k(x) = b_k$ for all indices k.

Suppose, conversely, that $\lim\limits_{x \to a} f_k(x) = b_k$ for all k. Let (x_n) be any sequence in $A \setminus \{a\}$ with $x_n \to a$. By the Sequential Characterization of Limits, we have $\lim\limits_{n \to \infty} f_k(x_n) = b_k$ for all k. By Limits of Component Sequences, we have $\lim\limits_{n \to \infty} f(x) = b$. By the Sequential Characterization of Limits again, it follows that $\lim\limits_{x \to a} f(x) = b$.

5.55 Theorem: *(Operations on Limits of Functions) Let $f, g : A \subseteq \mathbb{R}^\ell \to \mathbb{R}^m$, let $a \in A'$ and let $c \in \mathbb{R}$. Suppose that $\lim\limits_{x \to a} f(x) = u \in \mathbb{R}^m$ and $\lim\limits_{n \to \infty} g(x) = v \in \mathbb{R}^m$. Then*

(1) $\lim\limits_{x \to a} (f + g)(x) = u + v$,
(2) $\lim\limits_{x \to a} (cf)(x) = cu$,
(3) $\lim\limits_{x \to a} |f|(x) = |u|$,
(4) $\lim\limits_{x \to a} (f \cdot g)(x) = u \cdot v$, and
(5) when $m = 3$ we have $\lim\limits_{x \to \infty} (f \times g)(x) = u \times v$.

Proof: This follows from Operations on Limits of Sequences, together with the Sequential Characterization of Limits.

5.56 Theorem: *(Comparison Theorem) Let $f, g : A \subseteq \mathbb{R}^\ell \to \mathbb{R}$ with $f(x) \le g(x)$ for all $x \in A$ and let $a \in A'$.*

(1) If $\lim\limits_{x\to a} f(x) = u \in \mathbb{R} \cup \{\pm\infty\}$ and $\lim\limits_{x\to a} g(x) = v \in \mathbb{R} \cup \{\pm\infty\}$ then $u \le v$.

(2) If $\lim\limits_{x\to a} f(x) = \infty$ then $\lim\limits_{x\to a} g(x) = \infty$.

(3) If $\lim\limits_{x\to a} g(x) = -\infty$ then $\lim\limits_{x\to \infty} f(x) = -\infty$.

Proof: This follows from the Comparison Theorem for Sequences in $\mathbb{R}$ together with the Sequential Characterization of Limits.

5.57 Theorem: *(Squeeze Theorem) Let $f, g, h : A \subseteq \mathbb{R}^\ell \to \mathbb{R}$ with $f(x) \le g(x) \le h(x)$ for all $x \in A$, and let $u \in \mathbb{R} \cup \{\pm\infty\}$. If $\lim\limits_{x\to a} f(x) = u = \lim\limits_{x\to a} h(x)$ then $\lim\limits_{x\to a} g(x) = u$.*

Proof: This follows from the Squeeze Theorem for Sequences in $\mathbb{R}$ together with the Sequential Characterization of Limits.

Continuity

5.58 Definition: Let $A \subseteq \mathbb{R}^\ell$, let $B \subseteq \mathbb{R}^m$, and let $f : A \to B$. For $a \in A$, we say that f is **continuous at** a when

$$\forall\, \epsilon > 0\ \exists\, \delta > 0\ \forall x \in A\ \big(|x-a| < \delta \implies |f(x) - f(a)| < \epsilon\big).$$

We say that f is **continuous** (in A) when f is continuous at a for every $a \in A$. We say that f is **uniformly continuous** in A when

$$\forall\, \epsilon > 0\ \exists\, \delta > 0\ \forall\, a \in A\ \forall x \in A\ \big(|x-a| < \delta \implies |f(x) - f(a)| < \epsilon\big).$$

5.59 Theorem: *(Continuity and Limits) Let $A \subseteq \mathbb{R}^\ell$ and let $f : A \to \mathbb{R}^m$.*

(1) When a is a limit point of A, f is continuous at $a \iff \lim\limits_{x\to a} f(x) = f(a)$.

(2) When a is an isolated point of A, f is always continuous at a.

Proof: We leave the proof as an exercise.

5.60 Theorem: *(Sequential Characterization of Continuity) Let $A \subseteq \mathbb{R}^\ell$, let $f : A \to \mathbb{R}^m$, and let $a \in A$. Then f is continuous at a if and only if $\lim\limits_{n\to\infty} f(x_n) = f(a)$ for every sequence $(x_n)_{n \ge p}$ in A with $\lim\limits_{n\to\infty} x_n = a$.*

Proof: Suppose f is continuous at a. Let (x_n) be any sequence in A with $x_n \to a$. Let $\epsilon > 0$. Since f is continuous at a we can choose $\delta > 0$ so that $|x-a| < \delta \implies |f(x) - f(a)| < \epsilon$. Since $x_n \to a$ we can choose N so that $n \ge N \implies |x_n - a| < \delta$. Then for all $n \ge N$ we have $|x_n - a| < \delta$ hence $|f(x_n) - f(a)| < \epsilon$, and so $\lim\limits_{n\to\infty} f(x_n) = f(a)$, as required.

Suppose that f is not continuous at a. Choose $\epsilon > 0$ such that for every $\delta > 0$ there exists $x \in A$ such that $|x - a| < \delta$ and $|f(x) - f(a)| \ge \epsilon$. For each $n \in \mathbb{Z}^+$, choose $x_n \in A$ such that $|x_n - a| < \frac{1}{n}$ and $|f(x_n) - f(a)| \ge \epsilon$. Since $|x_n - a| < \frac{1}{n}$ for all $n \in \mathbb{Z}^+$ it follows that $x_n \to a$. Since $\big|f(x_n) - f(a)\big| \ge \epsilon$ for all n, it follows that $\lim\limits_{n\to\infty} f(x_n) \ne f(a)$. Thus we have found a sequence (x_n) in A with $x_n \to a$ such that $\lim\limits_{x\to a} f(x_n) \ne f(a)$.

5.61 Theorem: *(Local Determination of Continuity) Let $A \subseteq \mathbb{R}^\ell$, let $a \in A'$, and let $B = B^*(a, r) \cap A$ where $r > 0$. Let $f : A \to \mathbb{R}^m$ and $g : B \to \mathbb{R}^m$ and suppose that $f(x) = g(x)$ for all $x \in B$. Then f is continuous at a if and only if g is continuous at a.*

Proof: The proof is left as an exercise.

5.62 Theorem: *(Continuity of Component Functions) Let $A \subseteq \mathbb{R}^\ell$ and let $f : A \to \mathbb{R}^m$. Then f is continuous at a if and only if f_k is continuous at a for every index k.*

Proof: This can be proven by imitating the proof of Continuity of Component Sequences or by using the result of Continuity of Component Sequences together with the Sequential Characterization of Continuity.

5.63 Theorem: *(Operations on Continuous Functions) Let $A \subseteq \mathbb{R}^\ell$, let $f, g : A \to \mathbb{R}^m$, let $a \in A$ and let $c \in \mathbb{R}$. If f and g are continuous at a then so are each of the functions $f+g$, cf, $|f|$ and $f \cdot g$, and also $f \times g$ in the case that $m = 3$.*

Proof: This follows from the Sequential Characterization of Continuity along with Operations on Limits of Sequences.

5.64 Theorem: *(Composition and Limits) Let $f : A \subseteq \mathbb{R}^\ell \to \mathbb{R}^m$, let $g : B \subseteq \mathbb{R}^m \to \mathbb{R}^n$ and let $h = g \circ f : C \subseteq \mathbb{R}^\ell \to \mathbb{R}^n$ where $C = A \cap f^{-1}(B)$. Let $a \in C' \subseteq A'$ and let $b \in B'$. Suppose that $\lim\limits_{x \to a} f(x) = b$ and $\lim\limits_{y \to b} g(y) = c \in \mathbb{R}^n \cup \{\infty\}$.*

(1) If $f(x) \neq b$ for all $x \in C \setminus \{a\}$ then $\lim\limits_{x \to a} h(x) = c$.
(2) If $b \in B$ and g is continuous at b then $\lim\limits_{x \to a} h(x) = g(b) = c$.

Proof: We leave the proof of Part (1) as an exercise. To prove Part (2), suppose that $b \in B$ and g is continuous at b. Note that since $b \in B'$ and g is continuous at b we have $g(b) = \lim\limits_{y \to b} g(y) = c$ by Theorem 5.59. Let (x_k) be any sequence in $C \setminus \{a\}$ with $x_k \to a$. Since $C \subseteq A$, the sequence (x_k) also lies in $A \setminus \{a\}$. By the Sequential Characterization of Limits of Functions, since $\lim\limits_{x \to a} f(x) = b$ we have $\lim\limits_{k \to \infty} f(x_k) = b$. For each index k we have $x_k \in C = A \cap f^{-1}(B)$ so that $f(x_k) \in B$, and so the sequence $\big(f(x_k)\big)$ lies in B. By the Sequential Characterization of Continuity, since g is continuous at b and $f(x_k) \to b$ we have $\lim\limits_{k \to \infty} g\big(f(x_k)\big) = g(b) = c$, that is $\lim\limits_{k \to \infty} h(x_k) = g(b) = c$. By the Sequential Characterization of Limits, it follows that $\lim\limits_{x \to a} h(x) = g(b) = c$.

5.65 Corollary: *(Composition of Continuous Functions) Let $f : A \subseteq \mathbb{R}^\ell \to \mathbb{R}^m$, let $g : B \subseteq \mathbb{R}^m \to \mathbb{R}^n$, and let $h = g \circ f : C \subseteq \mathbb{R}^\ell \to \mathbb{R}^n$ where $C = A \cap f^{-1}(B)$.*

(1) If f is continuous at $a \in A$ and g is continuous at $b = f(a) \in B$ then h is continuous at a.
(2) If f is continuous in A and g is continuous in B then h is continuous in C.

5.66 Definition: An **elementary function** is a function $f : A \subseteq \mathbb{R}^\ell \to \mathbb{R}^m$ which can be obtained, using the operations of addition, subtraction, multiplication, division, and composition of functions (whenever those operations are defined) from the following functions, which we call the **basic elementary functions**: and the single-variable, real-valued functions c, x^n, $x^{1/n}$, e^x, $\ln x$, $\sin x$, $\cos x$, $\tan x$, $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$. and the k^{th} **inclusion map** $I_k : \mathbb{R} \to \mathbb{R}^\ell$ given by $I_k(t) = (0, \cdots, 0, t, 0, \cdots, 0) = t\, e_k$, and the k^{th} **projection map** $P_k : \mathbb{R}^\ell \to \mathbb{R}$ given by $P_k(x_1, \cdots, x_\ell) = x_k$.

5.67 Corollary: *Elementary functions are continuous in their domains.*

5.68 Exercise: Show that $\lim\limits_{(x,y) \to (0,0)} \dfrac{x^2 - 2y^2}{x^2 + y^2}$, $\lim\limits_{(x,y) \to (0,0)} \dfrac{xy}{x^2 + y^2}$ and $\lim\limits_{(x,y) \to (0,0)} \dfrac{xy^2}{x^2 + y^4}$ do not exist, and that $\lim\limits_{(x,y) \to (0,0)} \dfrac{3x^2 y}{x^2 + 2y^2} = 0$ and $\lim\limits_{(x,y) \to (0,0)} \dfrac{xy}{\sqrt{x^2 + y^2}} = 0$.

5.69 Theorem: *(Topological Characterization of Continuity) Let $A \subseteq \mathbb{R}^n$, let $B \subseteq \mathbb{R}^m$, and let $f : A \to B$.*

(1) f is continuous if and only if $f^{-1}(E)$ is open in A for every open set E in B.
(2) f is continuous if and only if $f^{-1}(F)$ is closed in A for every closed set F in B.

Proof: We prove Part (1) and leave the proof of Part (2) as an exercise. Suppose that f is continuous. Let E be an open set in B. Let $a \in f^{-1}(E)$ so we have $f(a) \in E$. Since $f(a) \in E$ and E is open in B we can choose $\epsilon > 0$ so that $B_B\big(f(a), \epsilon\big) \subseteq E$. Since f is continuous at a we can choose $\delta > 0$ so that for all $x \in A$, $|x - a| < \delta \Longrightarrow \big|f(x) - f(a)\big| < \epsilon$. Let $x \in B_A(a, \delta)$, that is let $x \in A$ with $|x - a| < \delta$. Since $x \in A$ and $f : A \to B$ we have $f(x) \in B$. Since $x \in A$ with $|x - a| < \delta$, we have and $\big|f(x) - f(a)\big| < \epsilon$. Since $f(x) \in B$ with $\big|f(x) = f(a)\big| < \epsilon$, we have $f(x) \in B_B\big(f(a), \epsilon\big) \subseteq E$ hence $x \in f^{-1}(E)$. Since $a \in B_A(a, \delta)$ was arbitrary, this shows that $B_A(a, \delta) \subseteq f^{-1}(E)$. Thus $f^{-1}(E)$ is open in A, as required.

Suppose, on the other hand, that $f^{-1}(E)$ is open in A for every open set E in B. Let $a \in A$ and let $\epsilon > 0$. The set $E = B_B\big(f(a), \epsilon\big)$ is open in B so the set $f^{-1}(E)$ is open in A, and so we can choose $\delta > 0$ such that $B_A(a, \delta) \subseteq f^{-1}(E)$. It follows that for all $x \in B_A(a, \delta)$ we have $f(a) \in E = B_B\big(f(a), \epsilon\big)$. Equivalently, for all $x \in A$, if $|x - a| < \delta$ then $f(x) \in B$ with $\big|f(x) - f(a)\big| < \epsilon$. hus f is continuous at a. Since $a \in A$ was arbitrary, f is continuous (in its domain A).

5.70 Theorem: *(Properties of Continuous Functions) Let $\emptyset \neq A \subseteq \mathbb{R}^n$, let $B \subseteq \mathbb{R}^m$, and let $f : A \to B$ be continuous.*

(1) If A is connected then $f(A)$ is connected.
(2) If A is compact then $f(A)$ is compact.
(3) If A is compact then f is uniformly continuous on A.
(4) If A is compact and $m = 1$ then $f(x)$ attains its maximum and minimum values on A.
(5) if A is compact and f is bijective then f^{-1} is continuous.

Proof: We sketch a proof for Parts (1), (2) and (4) and leave some details, along with the other two parts, as an exercise. To prove Part (1), suppose that $f(A)$ is disconnected. Choose open sets U and V in $\mathbb{R}^m$ which separate $f(A)$. Since f is continuous and U and V are open, it follows that $f^{-1}(U)$ and $f^{-1}(V)$ are open in A. Verify that $f^{-1}(U)$ and $f^{-1}(V)$ separate A, so A is disconnected.

To prove Part (2), suppose that A is compact. Let $S = \big\{U_k \big| k \in K\big\}$ be an open cover of $f(A)$ (with each U_k open in $\mathbb{R}^n$). For each set $k \in K$, since U_k is open in $\mathbb{R}^m$ and f is continuous, it follows that $f^{-1}(U_k)$ is open in A. Let $T = \big\{f^{-1}(U_k) \big| k \in K\big\}$. Verify that T is an open cover of A (with each set $f^{-1}(U_k)$ open in A). Since A is compact, we can choose a finite subset $J \subseteq K$ such that the set $\big\{f^{-1}(U_j) \big| j \in J\big\}$ is an open cover of A. Verify that the set $\big\{U_j \big| j \in J\big\}$ is an open cover for $f(A)$, so $f(A)$ is compact.

To prove Part (4), suppose that $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ with A is compact. Since A is compact and f is continuous, $f(A)$ is compact by Part (2). Since $f(A)$ is compact, it is closed and bounded by the Heine Borel Theorem. Since $f(A)$ is bounded and non-empty (since $A \neq \emptyset$) it has a supremum and an infemum in $\mathbb{R}$. Let $u = \sup f(A)$. By the Approximation Property of the Supremum, for each $k \in \mathbb{Z}^+$ we can choose $x_k \in A$ with $u - \frac{1}{n} < f(x_k) \leq u$, and it follows that $f(x_k) \to u$ and hence u is a limit point of $f(A)$. Since u is a limit point of $f(A)$ and $f(A)$ is closed, we have $u \in f(A)$. Thus we can choose

$a \in A$ such that $f(a) = u = \sup f(A) = \max f(A)$, and then f attains its maximum value at $a \in A$. Similarly, we can choose $b \in A$ such that $f(b) = \inf f(A) = \min f(A)$.

5.71 Definition: Let $A \subseteq \mathbb{R}^n$. For $a, b \in A$, the **line segment** between a and b is the set

$$[a, b] = \big\{a + t(b-a) \,\big|\, 0 \le t \le 1\big\}.$$

We say that A is **convex** when for every $a, b \in A$ we have $[a, b] \subseteq A$.

5.72 Example: Show that for $a \in \mathbb{R}^n$ and $r > 0$, the ball $B(a, r)$ is convex.

Proof: Let $b, c \in B(a, r)$ so we have $|b - a| < r$ and $|c - a| < r$. Let $x \in [b, c]$, say $x = b + t(c - b) = (1 - t)b + tc$ with $0 \le t \le 1$. Note that

$$x - a = (1 - t)b + tc - ((1 - t) + t)a = (1 - t)(b - a) + t(c - a).$$

By the Triangle Inequality, we have

$$\begin{aligned} |x - a| &= \big|(1 - t)(b - a) + t(c - a)\big| \le \big|(1 - t)(b - a)\big| + \big|t(c - a)\big| \\ &= (1 - t)|b - a| + t|c - a| < (1 - t)r + tr = r \end{aligned}$$

so that $x \in B(a, r)$. This shows that $[b, c] \subseteq B(a, r)$ and so $B(a, r)$ is convex.

5.73 Definition: Let $A \subseteq \mathbb{R}^n$ and let $a, b \in A$. A (continuous) **path** from a to b in A is a continuous function $f : [0, 1] \to A$ with $f(0) = a$ and $f(1) = b$. We say that A is **path-connected** when for every $a, b \in A$ there exists a continuous path from a to b in A.

5.74 Note: For $A \subseteq \mathbb{R}^n$, if A is convex then A is path connected because given $a, b \in A$, since $[a, b] \subseteq A$, the map $f(t) = a + t(b - a)$ is a continuous path from a to b in A.

5.75 Theorem: *(Path-Connectedness and Connectedness) Let $A \subseteq \mathbb{R}^n$.*
(1) If A is path-connected then A is connected.
(2) If A is open and connected then A is path-connected.

Proof: We prove Part (1) and leave Part (2) as an exercise. Suppose that A is path connected and suppose, for a contradiction, that A is not connected. Let U and V be open sets in $\mathbb{R}^n$ which separate A, that is $U \cap A \ne \emptyset$, $V \cap A \ne \emptyset$, $U \cap V = \emptyset$ and $A \subseteq U \cup V$. Choose $a \in U \cap A$ and $b \in V \cap A$. Since A is path connected we can choose a continuous path $f : [0, 1] \to A$ with $f(0) = a$ and $f(1) = b$. Since f is continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are open in $[0, 1]$. Since $f(0) = a \in U$ we have $0 \in f^{-1}(U)$ so $f^{-1}(U) \ne \emptyset$. Similarly $1 \in f^{-1}(V)$ so $f^{-1}(V) \ne \emptyset$. Since $U \cap V = \emptyset$ we also have $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ (indeed if we had $t \in f^{-1}(U) \cap f^{-1}(V)$ then we would have $f(t) \in U$ and $f(t) \in V$ so that $f(t) \in U \cap V$). Since $f : [0, 1] \to A \subseteq U \cup V$ it follows that $[0, 1] = f^{-1}(U) \cup f^{-1}(V)$ (indeed, given $t \in [0, 1]$ we have $f(t) \in A \subseteq U \cup V$, so either $f(t) \in U$ or $f(t) \in V$ hence either $t \in f^{-1}(U)$ or $t \in f^{-1}(V)$). Thus the open sets $f^{-1}(U)$ and $f^{-1}(V)$ separate $[0, 1]$. This is not possible since $[0, 1]$ is connected, so we have obtained the desired contradiction.

5.76 Example: Show that the set $U = \big\{(x, y) \in \mathbb{R}^2 \big| y > x^2\big\}$ is open in $\mathbb{R}^2$.

Solution: The map $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = y - x^2$ is continuous (it is an elementary function), and the interval $I = (0, \infty)$ is open and so the set $U = f^{-1}(I)$ is open (by the Topological Characterization of Continuity).

5.77 Example: Show that for $a \in \mathbb{R}^n$ and $r > 0$, the set $B(a, r)$ is connected.

Solution: Since $B(a, r)$ is convex (by Example 5.72), it is path connected (by Note 5.74), and hence it is connected (by Part 1 of Theorem 5.75).