PMATH 333 Real Analysis, Solutions to Assignment 4

1: (a) Let $A=\big\{(x,y)\in\mathbb{R}^2\,\big|\,4x^2+y^2<8x\big\}$. Prove, from the definition of an open set, that A is open.

Solution: First we remark that for all $(x,y)\in\mathbb{R}^2$ we have

$$(x,y)\in A \iff 4x^2+y^2<8x \iff 4(x-1)^2+y^2<4 \iff (x-1)^2+\big(\tfrac{y}{2}\big)^2 3<1 \iff \big(x,\tfrac{y}{2}\big)\in B\big((1,0),1\big).$$

Let $(a,b)\in A$. By the initial remark, we have $\big(a,\frac{b}{2}\big)\in B\big((1,0),1\big)$. Let $r=1-\big|\big(a,\frac{b}{2}\big)-(1,0)\big|$ and note that $r>0$. Then

$$\begin{aligned}(x,y)\in &B\big((a,b),r\big)\implies (x-a)^2+(y-b)^2<r^2\implies (x-a)^2+\tfrac14(y-b)^2<r^2\implies \big(x,\tfrac{y}{2}\big)\in B\big(\big(a,\tfrac{b}{2}\big),r\big)\\ &\implies \big|\big(x,\tfrac{y}{2}\big)-(1,0)\big|\le\big|\big(x,\tfrac{y}{2}\big)-\big(a,\tfrac{b}{2}\big)\big|+\big|\big(a,\tfrac{b}{2}\big)-(1,0)\big|<r+\big|\big(a,\tfrac{b}{2}\big)-(1,0)\big|=1\\ &\implies \big(x,\tfrac{y}{2}\big)\in B\big((1,0),1\big)\implies (x,y)\in A,\end{aligned}$$

with the final implication following from the initial remark. Thus $B\big((a,b),r\big)\subseteq A$, so A is open.

(b) Let $A=\big\{x\in\mathbb{R}^2\big|0<|x|\le 1\big\}$. Prove, from the definition of a compact set, that A is not compact.

Solution: For each $k\in\mathbb{Z}^+$ let U_k be the open set $U_k=\overline{B}\big(0,\frac1k\big)^c=\big\{x\in\mathbb{R}^n\big||x|>\frac1k\big\}$ and let $S=\big\{U_k\big|k\in\mathbb{Z}^+\big\}$. Note that $\bigcup S=\mathbb{R}^n\setminus\{0\}$ so S is an open cover of A. Let T be any finite subset of S. If $T=\emptyset$ then $\bigcup T=\emptyset$ so $A\not\subseteq\bigcup T$. Suppose that $T\neq\emptyset$, say $T=\big\{U_{k_1},U_{k_2},\cdots,U_{k_m}\big\}$ with $k_1<k_2<\cdots<k_m$. Since $U_{k_1}\subseteq U_{k_2}\subseteq\cdots\subseteq U_{k_m}$ we have $\bigcup T=\bigcup_{i=1}^m U_{k_i}=U_{k_m}=\overline{B}\big(0,\frac{1}{k_m}\big)^c$ and so $A\not\subseteq\bigcup T$. This shows that the open cover S has no finite subcover T, and so A is not compact.

(c) For $n\ge 1$, let $s_n=\sum\limits_{k=1}^n\big(\frac{1+i}{3}\big)^k$. Prove, from the definition of a limit, that $\lim\limits_{n\to\infty}s_n=\frac{1+3\,i}{5}$.

Solution: First note that $\mathbb{C}=\mathbb{R}^2$ (when $x,y\in\mathbb{R}$, the ordered pair $(x,y)\in\mathbb{R}$ is equal to the complex number $z=x+iy\in\mathbb{C}$), and the usual norm in $\mathbb{C}$ is equal to the usual norm in $\mathbb{R}^2$: for $z=x+iy=(x,y)$ we have $|z|=\sqrt{x^2+yy^2}=\big|(x,y)\big|$. From the formula for the sum of a geometric series, or by noting that

$$s_n\big(1-\tfrac{1+i}{3}\big)=\sum_{k=1}^n\big(\tfrac{1+i}{3}\big)^k-\sum_{k=2}^{n+1}\big(\tfrac{1+i}{3}\big)^k=\big(\tfrac{1+i}{3}\big)-\big(\tfrac{1+i}{3}\big)^{n+1},$$

we have

$$s_n=\frac{\big(\frac{1+i}{3}\big)-\big(\frac{1+i}{3}\big)^{n+1}}{1-\frac{1+i}{3}}=\frac{\big(\frac{1+i}{3}\big)\big(1-\big(\frac{1+i}{3}\big)^n\big)}{\frac{2-i}{3}}=\frac{(1+i)(2+i)\big(1-\big(\frac{1+i}{3}\big)^n\big)}{(2-i)(2+i)}=\tfrac{1+3i}{5}\Big(1-\big(\tfrac{1+i}{3}\big)^n\Big)=\tfrac{1+3i}{5}-\tfrac{1+3i}{5}\big(\tfrac{1+i}{3}\big)^n$$

and hence

$$\big|s_n-\tfrac{1+3i}{5}\big|=\big|\tfrac{1+3i}{5}\big(\tfrac{1+i}{3}\big)^n\big|=\big|\tfrac{1+3i}{5}\big|\,\big|\tfrac{1+i}{3}\big|^n=\tfrac{\sqrt{10}}{5}\big(\tfrac{\sqrt2}{3}\big)^n.$$

It follows that $\lim\limits_{n\to\infty}s_n=\frac{1+3i}{5}$: indeed given $\epsilon>0$, since $\frac{\sqrt2}{3}<1$ we can choose $m\in\mathbb{Z}^+$ so that $\big(\frac{\sqrt2}{3}\big)^m<\frac{\epsilon}{\sqrt{10}/5}$, and then when $n\ge m$ we have

$$\big|s_n-\tfrac{1+3i}{5}\big|=\tfrac{\sqrt{10}}{5}\big(\tfrac{\sqrt2}{3}\big)^n\le\tfrac{\sqrt{10}}{5}\big(\tfrac{\sqrt3}{2}\big)^m<\epsilon.$$

2: For sets $\emptyset \neq A, B \subseteq \mathbb{R}^n$, the *distance* between A and B is defined to be

$$d(A,B) = \inf\big\{|x-y| \,\big|\, x\in A, y\in B\big\}.$$

For a point $a \in \mathbb{R}^n$ and a set $\emptyset \neq B \subseteq \mathbb{R}^n$, the *distance* between a and B is defined to be

$$d(a,B) = d\big(\{a\},B\big) = \inf\big\{|a-y| \,\big|\, y\in B\big\}.$$

(a) Find nonempty closed sets $A, B \subseteq \mathbb{R}^2$ with $A \cap B = \emptyset$ such that $d(A,B) = 0$.

Solution: First we remark that for any non-empt sets A and B, since $|x-y| \geq 0$ for all $x \in A$ and $y \in B$, it follows that $d(A,B) = \inf\big\{|x-y| \,\big|\, x\in A\,,\ y\in B\big\} \geq 0$. Let A be the x-axis in $\mathbb{R}^2$, that is $A = \big\{(x,y)\in\mathbb{R}^2\big|y=0\big\}$ and let B be the graph of $y = \frac{1}{1+x^2}$ in $\mathbb{R}^2$, that is $B = \big\{(x,y)\in\mathbb{R}^2\big|\, y=\frac{1}{1+x^2}\big\}$. Note that A is closed in $\mathbb{R}^2$ since $A = f^{-1}\big(\{0\}\big)$ where $f: \mathbb{R}^2 \to \mathbb{R}$ is the continuous map given by $f(x,y) = y$, and B is closed in $\mathbb{R}^2$ since $B = g^{-1}\big(\{0\}\big)$ where $g: \mathbb{R}^2 \to \mathbb{R}$ is the continuous map given by $g(x,y) = \frac{1}{1+x^2}$. We claim that $d(A,B) = 0$. Let $\epsilon > 0$. Choose $x > 0$ such that $\frac{1}{1+x^2} < \epsilon$. Since $(x,0) \in A$ and $\big(x, \frac{1}{1+x^2}\big) \in B$, we have $d(A,B) \leq \big|(x,0) - (x, \frac{1}{1+x^2})\big| = \frac{1}{1+x^2} < \epsilon$. Since $0 \leq d(A,B) < \epsilon$ for every $\epsilon > 0$, we have $d(A,B) = 0$.

(b) Let $\emptyset \neq A, B \subseteq \mathbb{R}^n$ with A compact and B closed and $A \cap B = \emptyset$. Prove that $d(A,B) > 0$.

Solution: For each $a \in A$, since $B^c = \mathbb{R}^n \setminus B$ is open, we can choose $r_a > 0$ such that $B(a, r_a) \subseteq B^c$. Let $S = \big\{B(a,r_a) \,\big|\, a \in A\big\}$ and note that S is an open cover of A. Since A is compact, we can choose a finite sub-cover $T \subseteq S$, say $T = \big\{B(a_1, r_{a_1}, \cdots, B(a_\ell, r_{a_\ell})\big\}$. Let $r = \min\{r_{a_1}, \cdots, r_{a_\ell}\}$, and note that $r > 0$. We claim that $d(A,B) \geq r$. Let $x \in A$ and $y \in B$. Since $x \in A \subseteq \bigcup T = \bigcup_{k=1}^{\ell} B(a_k, r_{a_k})$, we can choose an index k such that $x \in B(a_k, r_{a_k})$, so $|x - a_k| < r_{a_k}$. Since $y \in B$ and $B(a_k, 2r_{a_k}) \subseteq B^c$, we have $y \notin B(a_k, 2r_{a_k})$, so $|y - a_k| \geq 2r_{a_k}$. Thus we have $2r_{a_k} \leq |y - a_k| \leq |y-x| + |x - a_k| < |y-x| + r_k$ and hence $|x-y| > r_{a_k} \geq r$. Since $|x-y| > r$ for all $x \in A$ and $y \in B$, it follows that $d(A,B) = \inf\big\{|x-y| \,\big|\, x\in A\,,\ y\in B\big\} \geq r$, as claimed.

(c) Fix a subset $\emptyset \neq B \subseteq \mathbb{R}^n$ and define $g: \mathbb{R}^n \to \mathbb{R}$ by $g(x) = d(x,B)$. Prove that $g(x)$ is uniformly continuous on $\mathbb{R}^n$ by showing that $\big|g(x) - g(y)\big| \leq d(x,y)$ for all $x, y \in \mathbb{R}^n$.

Solution: Suppose, for a contradiction, that we can choose $x, y \in \mathbb{R}^n$ with $\big|g(x) - g(y)\big| > d(x,y) = |x-y|$. Interchanging x and y if necessary, we suppose that $g(x) \geq g(y)$. Then we have $g(x) - g(y) > |x-y|$ so that $g(x) > g(y) + |x-y|$. Let $\epsilon = g(x) - \big(g(y) + |x-y|\big)$ so that we have $g(x) = g(y) + |x-y| + \epsilon$ with $\epsilon > 0$. Since $g(y) = \inf\big\{|y-b|\big|b \in B\big\}$, we can choose $b \in B$ so that $g(y) \leq |y-b| < g(y) + \epsilon$. Then we have $|x-b| \leq |x-y| + |y-b| < |x-y| + g(y) + \epsilon = g(x)$. But since $g(x) = \inf\big\{|x-b| \,\big|\, b \in B\big\}$ we must have $g(x) \leq |x-b|$, so we have obtained the desired contradiction.

3: For each of the following subsets $A \subseteq \mathbb{R}^n$, determine whether A is closed, whether A is compact, and whether A is connected.

(a) $A = \left\{ (t^2-1\,,\,t^3-t) \in \mathbb{R}^2 \,\middle|\, t \in \mathbb{R} \right\}$.

Solution: Note that $A = f(\mathbb{R}$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $(x,y) = f(t) = \big(t^2-1\,,\,t^3-t\big)$. Since $\mathbb{R}$ is connected and f is continuous, it follows that $A = f(\mathbb{R})$ is connected.

We claim that $A = g^{-1}(0)$ where $g : \mathbb{R}^2 \to \mathbb{R}$ is given by $g(x,y) = x^3 + x^2 - y^2$. Let $(x,y) \in A$, say $(x,y) = (t^2-1\,,\,t^3-t)$. Then $x^3 + x^2 = (t^6 - 3t^4 + 3t^2 - 1) + (t^4 - 2t^2 + 1) = t^6 - 2t^4 + t^2 = (t^3 - t)^2 = y^2$ so that $g(x,y) = 0$. This shows that $A \subseteq g^{-1}(0)$. Now let $(x,y) \in g^{-1}(0)$, so we have $y^2 = x^3 + x^2$. If $x = 0$ then $y^2 = x^3 + x^2 = 0$ so that $y = 0$, and in this case we can choose $t = 1$ to get $t^2 - 1 = 0 = x$ and $t^3 - t = 0 = y$ so that $(x,y) \in A$. If $x \neq 0$ then we can choose $t = \frac{y}{x}$ to get $t^2 - 1 = \frac{y^2}{x^2} - 1 = \frac{y^2 - x^2}{x^2} = \frac{x^3}{x^2} = x$ and $t^3 - t = t(t^2-1) = \frac{y}{x} \cdot x = y$ so that again $(x,y) \in A$. This shows that $g^{-1}(0) \subseteq A$, and hence $A = g^{-1}(0)$, as claimed. Since $\{0\}$ is closed and g is continuous, it follows that $A = g^{-1}(\{0\})$ is closed.

Finally, we note that A is not compact because A is not bounded: indeed given any $M \geq 0$ we can choose $t \geq 1$ such that $t^2 > M+1$, and then $(x,y) = (t^2-1\,,\,t^3-t) \in A$ with $\big|(x,y)\big| = \big|(t^2-1\,,\,t^3-t)\big| \geq t^2 - 1 > M$.

(b) $A = \left\{ (0,0) \neq (x,y) \in \mathbb{R}^2 \,\middle|\, \big|\mathrm{Re}\big(\frac{1}{x+iy}\big)\big| \geq 1 \right\}$ (where $\mathrm{Re}(z)$ denotes the real part of $z \in \mathbb{C}$).

Solution: For $a,b \in \mathbb{R}$ with $(a,b) \neq (0,0)$ we have $\frac{1}{a+ib} = \frac{a-ib}{a^2+b^2}$ so

$$\begin{aligned}\big|\mathrm{Re}\big(\tfrac{1}{a+ib}\big)\big| \geq 1 &\iff |a| \geq a^2 + b^2 \big(a^2 + b^2 \leq a \text{ or } a^2 + b^2 \leq -a\big) \\ &\iff \big(a - \tfrac{1}{2}\big)^2 + b^2 \leq \tfrac{1}{4} \text{ or } \big(a + \tfrac{1}{2}\big)^2 + b^2 \leq \tfrac{1}{4}.\end{aligned}$$

Thus $A = (B \cup C) \setminus \{(0,0)\}$ where B and C are the closed balls of radius $\frac{1}{2}$ centered at $\big(\frac{1}{2},0\big)$ and $\big(-\frac{1}{2},0\big)$. This set A is not closed since $(0,0) \notin A$ but $(0,0)$ is a limit point of A (indeed for $x_n = \big(\frac{1}{n},0\big)$ we have $x_n \in A$ and $x_n \to (0,0)$). Since A is not closed, it not compact. Also, A is not connected since it can be separated by the disjoint open sets $U = \big\{(x,y)\big|x > 0\big\}$ and $V = \big\{(x,y)\big|x < 0\big\}$.

(c) $A = \left\{ (x,y,z,w) \in \mathbb{R}^4 \,\middle|\, \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right)^2 = \left(\begin{smallmatrix} 3 & 2 \\ 4 & 3 \end{smallmatrix}\right) \right\}$.

Solution: Note that $\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right)^2 = \left(\begin{smallmatrix} x^2+yyz & xy+zw \\ xz+yw & yz+w^2 \end{smallmatrix}\right)$ and hence

$$(x,y,z,w) \in A \iff \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right)^2 = \left(\begin{smallmatrix} 3 & 2 \\ 4 & 3 \end{smallmatrix}\right) \iff \big(x^2 + yz = 3\,,\, xy + yw = 2\,,\, xz + zw = 4 \text{ and } yz + w^2 = 3\big).$$

Let $(x,y,z,w) \in A$. Since $x^2 + yz = 3 = yz + w^2$ we have $w^2 = x^2$ so that $w = \pm x$. We cannot have $w = -x$ because $w = -x \implies xy + yw = xy - yx = 0 \neq 2$, so we must have $w = x$. Since $w = x$, $xy + yw = 2$ gives $2xy = 2$ so that $\big(x \neq 0$ and $y = \frac{1}{x}\big)$, and $xz + zw = 4$ gives $2xz = 4$ so that $\big(x \neq 0$ and $z = \frac{2}{x}$. Since $w = x \neq 0$ and $y = \frac{1}{x}$ and $z = \frac{2}{x}$, $x^2 + yz = 3$ gives $x^2 + \frac{2}{x^2} = 3$ so that $x^4 - 3x^2 + 2 = 0$, that is $(x^2-1)(x^2-2) = 0$, and hence $x = \pm 1$ or $\pm\sqrt{2}$ so that $(x,y,z,w) = \pm(1,1,2,1)$ or $\pm\sqrt{2}(1,\frac{1}{2},1,1)$. This shows that $A \subseteq \big\{ \pm(1,1,2,1), \pm\sqrt{2}\,\big(1,\frac{1}{2},1,1\big)\big\}$. Conversely, if $(x,y,z,w) = \pm(1,1,2,1)$ or $\pm\sqrt{2}\,\big(1,\frac{1}{2},1,1\big)$, then we have $x^2 + yz = 3$, $xy + yw = 2$, $xz + xw = 4$ and $yz + w^2 = 3$ so that $(x,y,z,w) \in A$. Thus

$$A = \big\{ \pm(1,1,2,1), \pm\sqrt{2}\,\big(1,\tfrac{1}{2},1,1\big)\big\}.$$

Every finite set in $\mathbb{R}^n$ is closed (indeed the set $\{a_1, a_2, \cdots, a_\ell\}$ is the union of the ℓ closed sets $\{a_k\}$) and every finite set in $\mathbb{R}^n$ is bounded (indeed for every $x \in \{a_1, a_2, \cdots, a_\ell\}$ we have $|x| \leq M = \max\{|a_1|, |a_2|, \cdots, |a_\ell|\}$), and so A is closed and bounded, hence compact in $\mathbb{R}^4$. On the other hand, A is not connected because, for example, the open sets $U = \big\{(x,y,z,w) \in \mathbb{R}^4 \,\big|\, x<0\big\}$ and $V = \big\{(x,y,z,w) \in \mathbb{R}^4 \,\big|\, x>0\big\}$ separate A in $\mathbb{R}^4$.

4: (a) Prove that if the sets $A, B \subseteq \mathbb{R}^n$ are connected and $A \cap B \neq \emptyset$, then $A \cup B$ is connected.

Solution: Suppose that A and B are connected in $\mathbb{R}^n$ and that $A \cap B \neq \emptyset$. Choose $c \in A \cap B$. Suppose, for a contradiction, that $A \cup B$ is disconnected. Choose open sets U and V in $\mathbb{R}^n$ which separate $A \cup B$ (that is, $U \cap (A \cup B) \neq \emptyset$, $V \cap (A \cup B) \neq \emptyset$, $U \cup V = \emptyset$, and $A \cup B \subseteq U \cup V$). Since $c \in A \cap B \subseteq A \cup B \subseteq U \cup V$, either $c \in U$ or $c \in V$. By interchanging U and V if necessary, we can suppose that $c \in U$. Note that since $c \in A$ and $c \in U$ and A is connected, it follows that $A \subseteq U$ because if we had $A \not\subseteq U$ then (since $A \subseteq U \cup V$) we would have $A \cap V \neq \emptyset$, and then U and V would separate A (since $c \in U \cap A$ so $U \cap A \neq \emptyset$, and $U \cap V = \emptyset$, and $A \subseteq A \cup B \subseteq U \cup V$). Similarly, since $c \in B$ and $c \in U$ and B is connected, it follows that $B \subseteq U$. Since $A \subseteq U$ and $B \subseteq U$, we have $A \cup B \subseteq U$. Since $A \cup B \subseteq U$ and $U \cap V = \emptyset$, we must have $V \cap (A \cup B) = \emptyset$, which contradicts the fact that U and V separate $A \cup B$.

(b) Let A be the set of all $(a, b, c, d) \in \mathbb{R}^4$ such that the polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ has at least one repeated real root, and all of its (real or complex) roots lie in the closed unit ball $|z| \leq 1$. Prove that A is compact and connected.

Solution: A monic quartic polynomial with only real roots, at least one of which is repeated, is of the form

$$\begin{aligned} f(x) &= (x-r)(x-s)(x-t)^2 = (x^2 - (r+s)x + rs)(x^2 - 2tx + t^2) \\ &= x^4 - (2t + (r+s))x^3 + (t^2 + 2(r+s)t + rs)x^2 - ((r+s)t^2 + 2rst)x + rst^2 \end{aligned}$$

and a monic quartic polynomial with one repeated real root and a pair of conjugate complex roots is of the form

$$\begin{aligned} f(x) &= (x - (r+is))(x - (r-is))(x-t)^2 = (x^2 - 2rx + (r^2+s^2))(x^2 - 2tx + t^2) \\ &= x^4 - (2t + 2r)x^3 + (t^2 + 4rt + (r^2+s^2))x^2 - (2rt^2 + (r^2+s^2)t)x + (r^2+s^2)t^2). \end{aligned}$$

Thus we have $A = F(C) \cup G(D)$ where $F : C \subseteq \mathbb{R}^3 \to \mathbb{R}^4$ and $G : D \subseteq \mathbb{R}^3 \to \mathbb{R}^4$ given by

$$\begin{aligned} (a,b,c,d) &= F(r,s,t) = \big(2t + (r+s)\,,\, t^2 + 2(r+s)t + rs\,,\, (r+s)t^2 + 2rst\,,\, rst^2\big), \\ (a,b,c,d) &= G(r,s,t) = \big(-(2t+2r)\,,\, (t^2 + 4rt + (r^2+s^2))\,,\, -(2rt^2 + (r^2+s^2)t)\,,\, (r^2+s^2)t^2\big), \end{aligned}$$

where $C = \big\{(r,s,t) \in \mathbb{R}^3 \,\big|\, r,s,t \in [-1,1]\big\}$ and $D = \big\{(r,s,t) \in \mathbb{R}^3 \,\big|\, r^2+s^2 \leq 1\,,\, t \in [-1,1]\big\}$. The set C is closed because $C = f_1^{-1}\big([-1,1]\big) \cap f_2^{-1}\big([-1,1]\big) \cap f_3^{-1}\big([-1,1]\big)$ where $f_1(r,s,t) = r$, $f_2(r,s,t) = s$ and $f_3(r,s,t) = t$, and C is bounded because $(r,s,t) \in C \Longrightarrow r,s,t \in [-1,1] \Longrightarrow \big|(r,s,t)\big| = \sqrt{r^2+s^2+t^2} \leq \sqrt{1+1+1} = \sqrt{3}$. The set D is closed because $D = g_1^{-1}\big([0,1]\big) \cap g_2^{-1}\big([-1,1]\big)$ where $g_1(r,s,t) = r^2+s^2$ and $g_2(r,s,t) = t$, and D is bounded because $(r,s,t) \in D \Longrightarrow \big(r^2+s^2 \leq 1 \text{ and } t \in [-1,1]\big) \Longrightarrow \big|(r,s,t)\big| = \sqrt{r^2+s^2+t^2} \leq \sqrt{1+1} = \sqrt{2}$. Since C and D are closed and bounded, hence compact, and since F and G are continuous, it follows that $F(C)$ and $G(D)$ are compact, hence closed and bounded. Since $F(C)$ and $G(D)$ are closed, $A = F(C) \cup G(D)$ is closed. Since $F(C)$ and $F(D)$ are bounded, it follows that $A = F(C) \cup F(D)$ is bounded: indeed, if $F(C) \subseteq B(0,R)$ and $G(D) \subseteq B(0,S)$ then $F(C) \cup G(D) \subseteq B(0,T)$ where $T = \max(R,S)$. Since A is closed and bounded, it is compact.

We would also like to show that $f(C)$ and $f(D)$ are connected. To do this, let us first claim that if $E \subseteq \mathbb{R}^n$ and $F \subseteq \mathbb{R}^m$ are convex then $E \times F \subseteq \mathbb{R}^{n+m}$ is convex. Suppose E and F are convex. Let $a, c \in E$ and $b, d \in F$ so that $(a,b), (c,d) \in E \times F$. Let (x,y) be on the line segment from (a,b) to (c,d), say $(x,y) = (a,b) + t((c,d) - (a,b))$ where $0 \leq t \leq 1$. Then we have $x = a + t(c-a) \in [a,c] \subseteq E$ and $y = b + (d-b) \in [b,b] \subseteq F$ so that $(x,y) \in E \times F$. Thus $E \times F$ is convex, as claimed.

Since the sets $[-1,1]$ and $\overline{B}((0,0),1)$ are convex, it follows from the claim that $[-1,1] \times [-1,1]$ is convex, hence that $C = [-1,1] \times [-1,1] \times [-1,1]$ is convex, and also that $D = \overline{B}((0,0),1) \times [-1,1]$ is convex. Since C and D are convex, hence connected, and since F and G are continuous, the sets $F(C)$ and $G(D)$ are connected. Since $F(C)$ and $F(D)$ are connected and since $0 = (0,0,0,0) = F(0,0,0) = G(0,0,0)$ so that $0 \in F(C) \cap G(D)$, it follows from Part (a) that $A = F(C) \cup G(D)$ is connected.