PMATH 333 Real Analysis, Assignment 4 Due Mon Apr 8, 11:00 pm

1: (a) Let $A=\{(x,y)\in\mathbb{R}^2\,|\,4x^2+y^2<8x\}$. Prove, from the definition of an open set, that A is open.

(b) Let $A=\{x\in\mathbb{R}^2\,|\,0<|x|\leq 1\}$. Prove, from the definition of a compact set, that A is not compact.

(c) For $n\in\mathbb{Z}^+$, let $s_n=\sum\limits_{k=1}^{n}\left(\frac{1+i}{3}\right)^k$. Prove, from the definition of a limit, that $\lim\limits_{n\to\infty}s_n=\frac{1+3\,i}{5}$.

2: For sets $\emptyset\neq A,B\subseteq\mathbb{R}^n$, the *distance* between A and B is defined to be

$$d(A,B)=\inf\{|x-y|\,\big|\,x\in A, y\in B\}.$$

For a point $a\in\mathbb{R}^n$ and a set $\emptyset\neq B\subseteq\mathbb{R}^n$, the *distance* between a and B is defined to be

$$d(a,B)=d\big(\{a\},B\big)=\inf\{|a-y|\,\big|\,y\in B\}.$$

(a) Find nonempty closed sets $A,B\subseteq\mathbb{R}^2$ with $A\cap B=\emptyset$ such that $d(A,B)=0$.

(b) Let $\emptyset\neq A,B\subseteq\mathbb{R}^n$ with A compact and B closed and $A\cap B=\emptyset$. Prove that $d(A,B)>0$.

(c) Fix a subset $\emptyset\neq B\subseteq\mathbb{R}^n$ and define $g:\mathbb{R}^n\to\mathbb{R}$ by $g(x)=d(x,B)$. Prove that $g(x)$ is uniformly continuous on $\mathbb{R}^n$ by showing that $\big|g(x)-g(y)\big|\leq d(x,y)$ for all $x,y\in\mathbb{R}^n$.

3: For each of the following subsets $A\subseteq\mathbb{R}^n$, determine whether A is closed, whether A is compact, and whether A is connected.

(a) $A=\{(t^2-1\,,\,t^3-t)\in\mathbb{R}^2\,|\,t\in\mathbb{R}\}$.

(b) $A=\left\{(0,0)\neq(x,y)\in\mathbb{R}^2\,\middle|\,\left|\mathrm{Re}\left(\frac{1}{x+iy}\right)\right|\geq 1\right\}$ (where $\mathrm{Re}(z)$ denotes the real part of $z\in\mathbb{C}$).

(c) $A=\left\{(x,y,z,w)\in\mathbb{R}^4\,\middle|\,\begin{pmatrix}x&y\\z&w\end{pmatrix}^2=\begin{pmatrix}3&2\\4&3\end{pmatrix}\right\}$.

4: (a) Prove that if the sets $A,B\subseteq\mathbb{R}^n$ are connected and $A\cap B\neq\emptyset$, then $A\cup B$ is connected.

(b) Let A be the set of all $(a,b,c,d)\in\mathbb{R}^4$ such that the polynomial $f(x)=x^4+ax^3+bx^2+cx+d$ has at least one repeated real root, and all of its (real or complex) roots lie in the closed unit ball $|z|\leq 1$. Prove that A is compact and connected.