PMATH 333 Real Analysis, Solutions to Assignment 3

1: (a) Prove that there exist (at least) 3 distinct values of $x \in \mathbb{R}$ such that $8x^3 = 6x + 1$.

Solution: Let $f(x) = 8x^3 - 6x - 1$. Notice that $f(x)$ is continuous and we have $f(x) = 0 \iff 8x^3 = 6x + 1$. By the Intermediate Value Theorem, since $f(-1) = -3 < 0$ and $f\left(-\frac{1}{2}\right) = 1 > 0$, there is a number $x_1 \in \left(-1, -\frac{1}{2}\right)$ such that $f(x_1) = 0$. Similarly, since $f\left(-\frac{1}{2}\right) = 1 > 0$ and $f(0) = -1 < 0$, there is a number $x_2 \in \left(-\frac{1}{2}, 0\right)$ with $f(x_2) = 0$, and since $f(0) = -1 < 0$ and $f(1) = 1 > 0$, there is a number $x_3 \in (0,1)$ with $f(x_3) = 0$. $\big($In fact, the exact values of x_1, x_2 and x_3 are $x_1 = -\cos(40°)$, $x_2 = -\sin(10°)$ and $x_3 = \cos(20°)\big)$.

(b) Let $f : [0,2] \to \mathbb{R}$ be continuous with $f(0) = f(2)$. Prove that $f(x) = f(x+1)$ for some $x \in [0,1]$.

Solution: Define $g : [0,1] \to \mathbb{R}$ by $g(x) = f(x+1) - f(x)$. Note that g is continuous and

$$g(1) = f(2) - f(1) = f(0) - f(1) = -\big(f(1) - f(0)\big) = -g(0).$$

By the Intermediate Value Theorem, there is a number $x \in [0,1]$ with $g(x) = 0$ $\big($indeed if $g(0) \neq 0$ then one of the numbers $g(0)$ and $g(1)$ is positive and the other is negative so there is a number $x \in (0,1)$ with $g(x) = 0\big)$. Then we have $0 = g(x) = f(x+1) - f(x)$ and so $f(x) = f(x+1)$.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that $|f(x) - f(y)| \geq |x - y|$ for all $x, y \in \mathbb{R}$. Prove that f is bijective (that is, f is injective and surjective).

Solution: First we note that f is injective since when $x \neq y$ we have $|f(x) - f(y)| \geq |x - y| > 0$ so that $f(x) \neq f(y)$. Consider the two intervals $I = [0, \infty)$ and $J = (-\infty, 0]$. We claim that the image $f(I)$ entirely contains one of the two intervals $[f(0), \infty)$ and $(-\infty, f(0)]$. Since the set $\mathbb{Z}^+$ is infinite and f is injective, either there exist infinitely many $k \in \mathbb{Z}^+$ such that $f(k) > f(0)$ or there exist infinitely many $k \in \mathbb{Z}^+$ such that $f(k) < f(0)$. Consider the case that there exist infinitely many $k \in \mathbb{Z}^+$ such that $f(k) > f(0)$. We claim that, in this case, we have $[f(0), \infty) \subseteq f(I)$. Choose $k_1 < k_2 < k_3 < \cdots$ such that $f(k_j) > f(0)$ for every index j. For every index j, since $f(k_j) > f(0)$ and $\big|f(k_j) - f(0)\big| \geq |k_j - 0| = k_j$, we have $f(k_j) > f(0) + k_j$. Let $y \in [f(0), \infty)$. Choose j with $k_j \geq y + f(0)$ so that we have $f(k_j) \geq f(0) + k_j \geq y$. Since f is continuous and $f(0) \leq y \leq f(k_j)$, it follows from the Intermediate Value Theorem that we can choose $x \in [0, k_j]$ such that $f(x) = y$. This proves our claim that $[f(0), \infty) \subseteq f(I)$. Similarly, in the case that there exist infinitely many $k \in \mathbb{Z}^+$ with $f(k) < f(0)$ we have $(-\infty, f(0)] \subseteq f(I)$. Thus one of the two intervals $K = [f(0), \infty)$ and $L = (-\infty, f(0)]$ is entirely contained in $f(I)$. A similar argument shows that one of the two intervals K and L is entirely contained in $f(J)$. Since f is injective, it is not possible that one of K and L can be contained in both of $f(I)$ and $f(J)$ (for example if we had $K \subseteq f(I) \cap f(L)$, then given $f(0) \neq y \in K$ we could choose $0 \neq x_1 \in I$ and $0 \neq x_2 \in J$ with $f(x_1) = y = f(x_2)$). Thus K is contained in one of the sets $f(I)$ and $f(J)$, and L is contained in the other. Thus we have $\mathbb{R} = K \cup L \subseteq f(I) \cup f(J) = f(I \cup J) = f(\mathbb{R})$, or in other words, f is surjective.

2: (a) Find $\displaystyle\int_0^2 3x^2 - x\,dx$ by evaluating the limit of a sequence of Riemann sums.

Solution: For fixed $n \in \mathbb{Z}^+$, let $X_n = \{x_0, x_1, \cdots, x_n\}$ be the partition of $[0,2]$ into n equal-sized sub-intervals, so we have $x_k = \frac{2k}{n}$ with $\Delta_k x = \frac{2}{n}$, and for each index k, let t_k be the right endpoint, that is $t_k = x_k$, and let S_n be the resulting Riemann sum for the function $f(x) = 3x^2 - x$. Thus

$$\begin{aligned}\int_0^2 3x^2 - x\,dx &= \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^n f(t_k)\Delta_k x = \lim_{n\to\infty} \sum_{k=1}^n \left(3\left(\tfrac{2k}{n}\right)^2 - \tfrac{2k}{n}\right)\left(\tfrac{2}{n}\right)\\ &= \lim_{n\to\infty} \sum_{k=1}^n \left(\tfrac{24k^2}{n^3} - \tfrac{4k}{n^2}\right) = \lim_{n\to\infty}\left(\tfrac{24}{n^3}\sum_{k=1}^n k^2 - \tfrac{4}{n^2}\sum_{k=1}^n k\right)\\ &= \lim_{n\to\infty}\left(\tfrac{24}{n^3}\cdot\tfrac{n(n+1)(2n+1)}{6} - \tfrac{4}{n^2}\cdot\tfrac{n(n+1)}{2}\right) = 8 - 2 = 6\,.\end{aligned}$$

(b) Find $\displaystyle\int_0^4 \sqrt{x}\,dx$ by evaluating the limit of a sequence of Riemann sums.

Solution: Let $f(x) = \sqrt{x}$ on $[0,4]$. Note that the range of f is $[0,2]$. For $n \in \mathbb{Z}^+$, let $Y_n = \{y_0, y_1, \cdots, y_n\}$ be the partition of the range $[0,2]$ into n equal sub-intervals, so we have $y_k = \frac{2k}{n}$, let $X_n = \{x_0, x_1, \cdots, x_n\}$ be the corresponding partition of the domain $[0,4]$ given by $x_k = {y_k}^2 = \frac{4k^2}{n^2}$, and let $t_k = x_k$. Note that $\Delta_k x = (x_k - x_{k-1}) = \frac{4(k^2-(k-1)^2)}{n^2} = \frac{4(2k-1)}{n^2}$ and we have $|X_n| = \Delta_n x = \frac{4(2n-1)}{n^2} \to 0$ as $n \to \infty$, and so

$$\begin{aligned}\int_0^4 \sqrt{x}\,dx &= \lim_{n\to\infty}\sum_{k=1}^n f(t_k)\Delta_k x = \lim_{n\to\infty}\sum_{k=1}^n \sqrt{x_k}\,\Delta_k x = \lim_{n\to\infty}\sum_{k=1}^n \tfrac{2k}{n}\cdot\tfrac{4(2k-1)}{n^2}\\ &= \lim_{n\to\infty}\sum_{k=1}^n\left(\tfrac{16k^2}{n^3} - \tfrac{8k}{n^3}\right) = \lim_{n\to\infty}\left(\tfrac{16}{n^3}\sum_{k=1}^n k^2 - \tfrac{8}{n^3}\sum_{k=1}^n k\right)\\ &= \lim_{n\to\infty}\left(\tfrac{16}{n^3}\cdot\tfrac{n(n+1)(2n+1)}{6} - \tfrac{8}{n^3}\cdot\tfrac{n(n+1)}{2}\right) = \tfrac{16}{3}\,.\end{aligned}$$

3: (a) Define $f:[0,1]\to\mathbb{R}$ by $f(x)=x$ if $x\in\mathbb{Q}$, and $f(x)=2x$ if $x\notin\mathbb{Q}$. Prove that f is not integrable on $[0,1]$.

Solution: Let $g(x)=x$ and $h(x)=2x$. Note that g and h are both integrable on $[0,1]$ with $\int_0^1 g=\int_0^1 x\,dx=\frac{1}{2}$ and $\int_0^1 h=\int_0^1 2x\,dx=1$. Suppose, for a contradiction, that f is integrable on $[0,1]$ and let $I=\int_0^1 f$. Taking $\epsilon=\frac{1}{8}$, choose $\delta>0$ such that for all partitions X of $[0,1]$ with $|X|<\delta$, we have $|F-I|<\frac{1}{8}$, $\left|G-\frac{1}{2}\right|<\frac{1}{8}$ and $\left|H-1\right|<\frac{1}{8}$ for all Riemann sums F, G and H for the functions f, g and h (respectively) on the partition X. Choose a partition $X=\{x_0,x_1,\cdots,x_n\}$ of $[0,1]$ with $|X|<\delta$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can choose sample points $t_k\in[x_{k-1},x_k]$ with $t_k\in\mathbb{Q}$, and then we have $f(t_k)=t_k=g(t_k)$ for all k. Thus $S=\sum_{k=1}^n t_k\,\Delta_k x$ is, simultaneously, a Riemann sum for both f and g on X, so we have $|S-I|<\frac{1}{8}$ and $|S-\frac{1}{2}|<\frac{1}{8}$, and hence $\left|I-\frac{1}{2}\right|<\frac{1}{4}$ so that $I<\frac{3}{4}$. Since $\mathbb{R}\setminus\mathbb{Q}$ is dense in $\mathbb{R}$, we can choose sample points $s_k\in[x_{k-1},x_k]$ with $s_k\notin\mathbb{Q}$, and then we have $f(s_k)=2s_k=h(s_k)$ for all k. Thus $T=\sum_{k=1}^n 2s_k\,\Delta_k x$ is, simultaneously, a Riemann sum for both f and h on X, so we have $|T-I|<\frac{1}{8}$ and $|T-1|<\frac{1}{8}$, and hence $|I-1|<\frac{1}{4}$ so that $I>\frac{3}{4}$.

(b) Define $g:[0,1]\to\mathbb{R}$ by $g\left(\frac{1}{n}\right)=1$ for each $n\in\mathbb{Z}^+$, and $g(x)=0$ when $x\notin\left\{\frac{1}{n}\,\middle|\,n\in\mathbb{Z}^+\right\}$. Determine (with proof) whether g is integrable on $[0,1]$.

Solution: We claim that g is integrable. Let $\epsilon>0$. Choose $n\in\mathbb{Z}^+$ such that $\frac{1}{n}<\frac{\epsilon}{2}$. Choose $\delta>0$ small enough so that $\frac{1}{n}+\delta<\frac{1}{n-1}-\delta$ and so that $2(n-1)\delta<\frac{\epsilon}{2}$. Let $X=\{x_0,x_1,\cdots,x_{2n}\}$ be the partition of $[0,1]$ given by

$$x_0=0\,,\ x_1=\tfrac{1}{n}\,,\ x_2=\tfrac{1}{n}+\delta\,,\ x_3=\tfrac{1}{n-1}-\delta\,,\ x_4=\tfrac{1}{n-1}+\delta\,,\ x_5=\tfrac{1}{n-2}-\delta\,,\ \cdots\ ,\ x_{2n-1}=1-\delta\,,\ x_{2n}=1$$

so for $1<k<2n$, when k is odd we have $x_k=\frac{1}{n-\frac{k-1}{2}}-\delta$ and when k is even we have $x_k=\frac{1}{n-\frac{k-2}{2}}+\delta$ (note that we chose δ small enough so that $\frac{1}{n}+\delta<\frac{1}{n-1}-\delta$ to ensure that the endpoints x_k are in increasing order). Let M_k and m_k denote the supremum and the infimum of $g(t)$ for $t\in[x_{k-1},x_k]$. In the first interval $[x_0,x_1]=\left[0,\frac{1}{n}\right]$, we have $M_1=1$ and $m_1=0$ and $\Delta_1 x=\frac{1}{n}$ so that $(M_1-m_1)\Delta_1 x=\frac{1}{n}$. In the second interval $[x_1,x_2]=\left[\frac{1}{n},\frac{1}{n}+\delta\right]$, we have $M_2=1$ and $m_2=0$ and $\Delta_2 x=\delta$ so that $(M_2-m_2)\Delta_2 x=\delta$. When k is odd with $2<k<2n$ and $\ell=n-\frac{k-1}{2}$, the k^{th} interval is $[x_{k-1},x_k]=\left[\frac{1}{\ell}-\delta,\frac{1}{\ell}+\delta\right]$ and we have $M_k=1$, $m_k=0$ and $\Delta_k x=2\delta$ so that $(M_k-m_k)\Delta_k x=2\delta$. When k is even with $2<k<2n$ and $\ell=n-\frac{k-2}{2}$, the k^{th} interval is $[x_{k-1},x_k]=\left[\frac{1}{\ell+1}+\delta,\frac{1}{\ell}-\delta\right]$ and we have $M_k=m_k=0$ so that $(M_k-m_k)\Delta_k x=0$. Finally, in the $2n^{\text{th}}$ interval $[x_{2n-1},x_{2n}]=\left[1-\frac{1}{\delta},1\right]$, we have $M_{2n}=1$, $m_{2n}=0$ and $\Delta_{2n}x=\delta$ so that $(M_{2n}-m_{2n})\Delta_{2n}x=\delta$. Thus for this partition, we have

$$U(g,X)-L(g,X)=\sum_{k=1}^{2n}(M_k-m_k)\Delta_k x=\tfrac{1}{n}+\delta+(n-2)2\delta+\delta=\tfrac{1}{n}+2(n-1)\delta<\tfrac{\epsilon}{2}+\tfrac{\epsilon}{2}=\epsilon.$$

4: Determine (with proof) which of the following statements are true (for all functions).

(a) Let f be bounded on $[a,b]$, let $X_n=\{x_{n,0},x_{n,1},\cdots,x_{n,n}\}$ be the partition of $[a,b]$ into n equal subintervals, and let $S_n=\sum_{i=1}^{n}f(x_{n,i})\Delta_{n,i}x$. If $\lim_{n\to\infty}S_n$ exists and is finite, then f is integrable on $[a,b]$.

Solution: This is FALSE. Consider the function $f:[0,1]\to[0,1]$ given by $f(x)=1$ when $x\in\mathbb{Q}$ and $f(x)=0$ when $x\notin\mathbb{Q}$. We have seen that this function is not integrable. But for the partition X_n of $[0,1]$ into n equal-sized subintervals, the endpoints $x_{n,k}=\frac{k}{n}$ all lie in $\mathbb{Q}$ so that we always have $f(x_{n,k})=1$ for all n,k. It follows that for all $n\in\mathbb{Z}^+$ we have $S_n=\sum\limits_{k=1}^{n}f(x_{n,k})\Delta_k x=\sum\limits_{k=1}^{n}1\cdot\frac{1}{n}=1$, and so $\lim\limits_{n\to\infty}S_n=1$.

(b) If $f\le g\le h$ on $[a,b]$ and f and h are integrable on $[a,b]$ with $\int_a^b f=\int_a^b h$, then g is integrable on $[a,b]$.

Solution: This is TRUE. Suppose f and h are integrable on $[a,b]$ with equal integrals, say $I=\int_a^b f=\int_a^b g$. Let $\epsilon>0$. Choose partitions X_1 and X_2 of $[a,b]$ so that $U(f,X_1)-L(f,X_1)<\frac{\epsilon}{2}$ and $U(h,X_2)-L(h,X_2)<\frac{\epsilon}{2}$. Let $X=X_1\cup X_2$. Since $X_1\subseteq X$ we have $L(f,X_1)\le L(f,X)\le I\le U(f,X)\le U(f,X_1)$ so that $0\le I-L(f,X)\le U(f,X_1)-L(f,X_1)<\frac{\epsilon}{2}$. Similarly, since $X_2\subseteq X$ we have $0<U(h,X)-I<\frac{\epsilon}{2}$. Say $X=(x_0,x_1,\cdots,x_n)$. For each index k with $1\le k\le n$, let $M_k(f)=\sup\big\{f(t)\big|t\in[x_{k-1},x_k]\big\}$ and $m_k(f)=\inf\big\{f(t)\big|t\in[x_{k-1},x_k]\big\}$, and define $M_k(g)$, $m_k(g)$, $M_k(h)$ and $m_k(h)$ similarly. Since $f(t)\le g(t)\le h(t)$ for all $t\in[a,b]$, it folows that $M_k(f)\le M_k(g)\le M_k(h)$ and $m_k(f)\le m_k(g)\le m_k(h)$ for all indices k. Since $M_k(g)\le M_k(h)$ for all k we have $U(g,X)=\sum\limits_{k=1}^{n}M_k(g)\Delta_k x\le\sum\limits_{k=1}^{n}M_k(h)\Delta_k x=U(h,X)$, Similarly, since $m_k(f)\le m_k(g)$ for all k we have $L(f,X)\le L(g,X)$. Thus

$$U(g,X)-L(g,X)\le U(h,X)-L(f,X)=\big(U(h,X)-I\big)+\big(I-L(f,X)\big)<\tfrac{\epsilon}{2}+\tfrac{\epsilon}{2}=\epsilon.$$

(c) For $f:[0,\infty)\to\mathbb{R}$, we say that f is *improperly integrable* on $[0,\infty)$ when f is integrable on $[0,r]$ for all $r>0$ and $\lim\limits_{r\to\infty}\int_0^r f(x)\,dx$ exists as a finite real number. If f is improperly integrable on $[0,\infty)$ then so is f^2.

Solution: This is FALSE. For example, define $f:[0,\infty)\to\mathbb{R}$ by $f(x)=\frac{(-1)^{n+1}}{\sqrt{n}}$ for $x\in[n-1,n)$ with $n\in\mathbb{Z}^+$. We claim that f is improperly integrable on $[0,\infty)$. Note that f is integrable on $[0,r]$ for all $r>0$ since it is integrable on each interval $[k-1,k]$ with $1\le k\le r$ and on the interval $[n,r]$ with $n=\lfloor r\rfloor$. Also note that $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}}$ converges by the Alternating Series test. Let $S_n=\sum_{k=1}^{n}\frac{(-1)^{k+1}}{\sqrt{k}}$ and let $S=\lim\limits_{n\to\infty}S_n$ (in fact, $S=\ln 2$, but we do not need to prove this). We claim that $\lim\limits_{r\to\infty}\int_0^r f=S$. Let $\epsilon>0$. Choose $m\in\mathbb{Z}^+$ so that when $n\ge m$ we have $|S_n-S|<\frac{\epsilon}{2}$ and $\frac{1}{\sqrt{n+1}}<\frac{\epsilon}{2}$. Let $r\ge m$, let $n=\lfloor r\rfloor$, and note that $n\ge m$. Then

$$\left|\int_0^r f-S\right|=\left|\int_0^n f+\int_n^r f-S\right|=\left|S_n+\tfrac{(-1)^n(r-n)}{\sqrt{n+1}}-S\right|\le|S_n-S|+\tfrac{1}{\sqrt{n+1}}<\epsilon\,.$$

Thus $\int_0^r=S$, as claimed, so that f is improperly integrable on $[0,\infty)$.

On the other hand, the map $f^2:[0,\infty)\to\mathbb{R}$ is given by $f^2(x)=\frac{1}{n}$ for $x\in[n-1,n]$ with $n\in\mathbb{Z}^+$, and the harmonic series $\sum\frac{1}{n}$ diverges. Given $R\ge 0$ we can choose $m\in\mathbb{Z}^+$ so that $n\ge m\implies\sum_{k=1}^{n}\frac{1}{k}>R$, and then for $r\ge m$, and letting $n=\lfloor r\rfloor$, we have

$$\int_0^r f^2=\int_0^n f^2+\int_n^r f^2\ge\int_0^n f^2=\sum_{k=1}^{n}\tfrac{1}{k}>R.$$

Thus $\lim\limits_{r\to\infty}\int_0^r f=\infty$ so that f^2 is not improperly integrable on $[0,\infty)$.