PMATH 333 Real Analysis, Assignment 3 Due Fri Mar 8, 11:00 pm

1: (a) Prove that there exist (at least) 3 distinct values of $x \in \mathbb{R}$ such that $8x^3 = 6x + 1$.

(b) Let $f : [0, 2] \to \mathbb{R}$ be continuous with $f(0) = f(2)$. Prove that $f(x) = f(x + 1)$ for some $x \in [0, 1]$.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that $|f(x) - f(y)| \geq |x - y|$ for all $x, y \in \mathbb{R}$. Prove that f is bijective (that is, f is injective and surjective).

2: (a) Find $\displaystyle\int_0^2 3x^2 - x\,dx$ by evaluating the limit of a sequence of Riemann sums for the function $f(x) = 3x^2 - x$ using partitions of $[0, 2]$ into equal-sized subintervals.

(b) Find $\displaystyle\int_0^4 \sqrt{x}\,dx$ by evaluating the limit of a sequence of Riemann sums for the function $f(x) = \sqrt{x}$ using suitable partitions of $[0, 4]$.

3: (a) Define $f : [0, 1] \to \mathbb{R}$ by $f(x) = x$ if $x \in \mathbb{Q}$, and $f(x) = 2x$ if $x \notin \mathbb{Q}$. Prove that f is not integrable on $[0, 1]$.

(b) Define $g : [0, 1] \to \mathbb{R}$ by $g\left(\frac{1}{n}\right) = 1$ for each $n \in \mathbb{Z}^+$, and $g(x) = 0$ when $x \notin \left\{\frac{1}{n} \middle| n \in \mathbb{Z}^+\right\}$. Determine (with proof) whether g is integrable on $[0, 1]$.

4: Determine (with proof) which of the following statements are true (for all functions).

(a) Let f be bounded on $[a, b]$, let $X_n = \{x_{n,0}, x_{n,1}, \cdots, x_{n,n}\}$ be the partition of $[a, b]$ into n equal subintervals, and let $S_n = \displaystyle\sum_{i=1}^{n} f(x_{n,i})\Delta_{n,i}x$. If $\displaystyle\lim_{n\to\infty} S_n$ exists and is finite, then f is integrable on $[a, b]$.

(b) If $f \leq g \leq h$ on $[a, b]$ and f and h are integrable on $[a, b]$ with $\displaystyle\int_a^b f = \int_a^b h$, then g is integrable on $[a, b]$.

(c) For $f : [0, \infty) \to \mathbb{R}$, we say that f is *improperly integrable* on $[0, \infty)$ when f is integrable on $[0, r]$ for all $r > 0$ and $\displaystyle\lim_{r\to\infty} \int_0^r f(x)\,dx$ exists as a finite real number. If f is improperly integrable on $[0, \infty)$ then so is f^2.