PMATH 333 Real Analysis, Solutions to Assignment 3.5

1: For each of the following sequences of functions $(f_n)_{n\geq 1}$, find the set A of points $x\in\mathbb{R}$ for which the sequence of real numbers $\big(f_n(x)\big)_{n\geq 1}$ converges, find the pointwise limit $g(x)=\lim\limits_{n\to\infty} f_n(x)$ for all $x\in A$, and determine whether $f_n\to g$ uniformly in A.

(a) $f_n(x)=(\sin x)^n$

Solution: If $x=\frac{\pi}{2}+2\pi k$ for some $k\in\mathbb{Z}$ then $\sin x=1$ and so $f_n(x)=1$ for all n, and so $\lim_{n\to\infty} f_n(x)=1$. If $x=-\frac{\pi}{2}+2\pi k$ for some $k\in\mathbb{Z}$ then $\sin x=-1$ so $f_n(x)=(-1)^n$ and so $\lim\limits_{n\to\infty} f_n(x)$ does not exist. If $x\neq\frac{\pi}{2}+\pi k$ for any $k\in\mathbb{Z}$ then $|\sin x|<1$ so $\lim\limits_{n\to\infty} f_n(x)=\lim\limits_{n\to\infty}(\sin x)^n=0$. Thus the set of points for which $(f_n(x))$ converges is $A=\big\{x\in\mathbb{R}\,\big|\,x\neq-\frac{\pi}{2}+2\pi k \text{ for any } k\in\mathbb{Z}\big\}$, and the limit function $g:A\to\mathbb{R}$ is given by

$$g(x)=\begin{cases}0\ , \text{ if } x\neq\frac{\pi}{2}+\pi k \text{ for any } k\in\mathbb{Z}\\ 1\ , \text{ if } x=\frac{\pi}{2}+2\pi k \text{ for some } k\in\mathbb{Z}.\end{cases}$$

Since each function $f_n(x)$ is continuous everywhere but $g(x)$ is not continuous at the points $x=\frac{\pi}{2}+2\pi k$ with $k\in\mathbb{Z}$, the convergence cannot be uniform.

(b) $f_n(x)=x\,e^{-nx^2}$

Solution: When $x=0$ we have $f_n(x)=0$ for all n, and when $x\neq 0$ we have $\lim_{n\to\infty} nx^2=\infty$ so that $\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty} x\,e^{-nx^2}=0$. Thus the set of points at which $(f_n)_{n\geq 1}$ converges is $A=\mathbb{R}$, and the limit function is the zero function $g(x)=\lim\limits_{n\to\infty} f_n(x)=0$. We claim that $f_n\to 0$ uniformly on $\mathbb{R}$. Since $e^{-nx^2}>0$ for all x, we have $f_n(x)\leq 0$ when $x\leq 0$ and $f_n(x)\geq 0$ when $x\geq 0$. Also, we have $f_n{}'(x)=(1-2nx^2)e^{-nx^2}$ so that $f_n'(x)\leq 0$ when $x\leq-\frac{1}{\sqrt{2n}}$, $f_n'(x)\geq 0$ when $-\frac{1}{\sqrt{2n}}\leq x\leq\frac{1}{\sqrt{2n}}$, and $f_n'(x)\leq 0$ when $x\geq\frac{1}{\sqrt{2n}}$. Thus the absolute minimum value of f_n is $f\big(-\frac{1}{\sqrt{2n}}\big)=-\frac{1}{\sqrt{2ne}}$ and the absolute maximum value of f_n is $f_n\big(\frac{1}{\sqrt{2n}}\big)=\frac{1}{\sqrt{2ne}}$, and hence $|f_n(x)-0|<\frac{1}{\sqrt{2ne}}$ for all $x\in\mathbb{R}$. It follows that $f_n\to 0$ uniformly on $\mathbb{R}$ as claimed. To be explicit, given $\epsilon>0$ we can choose $m\in\mathbb{Z}^+$ with $m>\frac{1}{2e\,\epsilon^2}$ and then when $n\geq m$ we have $n>\frac{1}{2e\,\epsilon^2}$ so that $\frac{1}{\sqrt{2ne}}<\epsilon$, and hence $|f_n(x)-0|<\frac{1}{\sqrt{2n\epsilon}}<\epsilon$ for all $x\in\mathbb{R}$.

(c) $f_n(x)=x^n-x^{2n}$

Solution: Note that $f_n(x)=x^n-x^{2n}=x^n(1-x^2)$. When $x<-1$, for even values of n we have $x^n\to+\infty$ and $(1-x^n)\to-\infty$ so that $f_n(x)=x^n(1-x^n)\to-\infty$, and for odd values of n we have $x^n\to-\infty$ and $(1-x^2)\to+\infty$ so that $f_n(x)\to-\infty$, and so $\lim\limits_{n\to\infty} f_n(x)=-\infty$. When $x=-1$, for even values of n we have $f_n(x)=x^n-x^{2n}=1-1=0$ and for odd values of n we have $f_n=x^n-x^{2n}=-1-1=-2$ and so $\lim\limits_{n\to\infty} f_n(x)$ does not exist. When $-1<x<1$ we have $x^n\to 0$ and $x^{2n}\to 0$ and so $\lim\limits_{n\to\infty} f_n(x)=0$. When $x=1$ we have $f_n(x)=0$ for all n so $\lim\limits_{n\to\infty} f_n(x)=0$ When $x>1$ we have $x^n\to\infty$ and $(1-x^n)\to-\infty$ and so $f_n(x)=x^n(1-x^n)\to-\infty$. Thus the set of points $x\in\mathbb{R}$ for which the sequence $\big(f_n(x)\big)$ converges is $A=(-1,1]$ and the limit function $g:(-1,1]\to\mathbb{R}$ is given by $g(x)=0$ for all $x\in(-1,1]$. The convergence is not uniform because given any $n\in\mathbb{Z}^+$, since f_n is continuous everywhere with $f_n(-1)=-2$ and $f_n(0)=0$ we can, by the Intermediate Value Theorem, choose $x\in(-1,0)$ such that $f_n(x)=-1$ and then we have $|f_n(x)-g(x)|=1$.

2: Let $(a_n)_{n\geq 1}$ be a sequence in $\mathbb{R}$, let $(f_n)_{n\geq 1}$ be a sequence of functions $f_n : A \subseteq \mathbb{R} \to \mathbb{R}$, let $g : A \subseteq \mathbb{R} \to \mathbb{R}$ and let $h : \mathbb{R} \to \mathbb{R}$.

(a) Suppose that $\sum_{n\geq 1} a_n$ converges and $\big|f_{n+1}(x) - f_n(x)\big| \leq a_n$ for all $n \geq 1$ and all $x \in A$. Show that $(f_n)_{n\geq 0}$ converges uniformly on A.

Solution: Let $\epsilon > 0$. Since each $a_n \geq 0$ and $\sum a_n$ converges, by the Cauchy Criterion for Series we can choose $m \in \mathbb{Z}^+$ such that for all $\ell > k \geq m$ we have $\sum\limits_{n=k+1}^{\ell} a_n < \epsilon$. Then for all $\ell > k \geq m$ and all $x \in A$ we have

$$\begin{aligned}\big|f_\ell(x) - f_k(x)\big| &= \big|(f_\ell(x) - f_{\ell-1}(x)) + (f_{\ell-1}(x) - f_{\ell-2}(x)) + \cdots + (f_{k+1}(x) - f_k(x))\big| \\ &\leq \big|f_\ell(x) - f_{\ell-1}(x)\big| + \big|f_{\ell-1}(x) - f_{\ell-2}(x)\big| + \cdots + \big|f_{k+1}(x) - f_k(x)\big| \\ &\leq a_\ell + a_{\ell-1} + \cdots + a_{k+1} = \sum_{n=k+1}^{\ell} a_n < \epsilon.\end{aligned}$$

Thus $f_n \to f$ uniformly in A by the Cauchy Criterion for Uniform Convergence of Sequences of Functions.

(b) Suppose that $f_n \to g$ uniformly on A and $f_n(x) \geq 0$ for all $n \geq 1$ and all $x \in A$. Show that $\sqrt{f_n} \to \sqrt{g}$ uniformly on A.

Solution: Let $\epsilon > 0$. Since $f_n \to g$ uniformly on A we can choose $m \in \mathbb{Z}^+$ such that for all $n \in \mathbb{Z}^+$, if $n \geq m$ then $|f_n(x) - g(x)| < \epsilon^2$ for all $x \in A$. Let $n \in \mathbb{Z}^+$ with $n \geq m$ and let $x \in A$. If $\sqrt{f_n(x)} + \sqrt{g(x)} < \epsilon$ then (by the Triangle Inequality) $\big|\sqrt{f_n(x)} - \sqrt{g(x)}\big| \leq \sqrt{f_n(x)} + \sqrt{g(x)} < \epsilon$, and if $\sqrt{f_n(x)} + \sqrt{g(x)} \geq \epsilon$ then

$$\big|\sqrt{f_n(x)} - \sqrt{g(x)}\big| = \frac{\big|\sqrt{f_n(x)} - \sqrt{g(x)}\big|\,\big|\sqrt{f_n(x)} + \sqrt{g(x)}\big|}{\big|\sqrt{f_n(x)} + \sqrt{g(x)}\big|} = \frac{|f_n(x) - g(x)|}{\sqrt{f_n(x)} + \sqrt{g(x)}} < \frac{\epsilon^2}{\epsilon} = \epsilon.$$

Thus $\sqrt{f_n} \to \sqrt{g}$ uniformly on A, as required.

(c) Suppose that $f_n \to g$ uniformly on A, g is bounded, and h is continuous. Prove that $h \circ f_n \to h \circ g$ uniformly on A.

Solution: Since f is bounded we can choose $M \geq 0$ so that $|f(x)| \leq M$ for all $x \in A$. Since $f_n \to f$ uniformly on A we can choose $m_1 \in \mathbb{Z}^+$ such that $n \geq m_1 \Longrightarrow \big|f_n(x) - f(x)\big| \leq 1$ for all $x \in A$. Then for $n \geq m_1$ and $x \in A$ we have $\big|f_n(x)\big| \leq \big|f_n(x) - f(x)\big| + \big|f(x)\big| \leq 1 + M$ so that $f_n(x) \in \big[-(M+1), M+1\big]$. Let $\epsilon > 0$. Since g is uniformly continuous on $\big[-(M+1), M+1\big]$, we can choose $\delta > 0$ so that for all $u, v \in \big[-(M+1), M+1\big]$ we have $|u - v| < \delta \Longrightarrow \big|g(u) - g(v)\big| < \epsilon$. Since $f_n \to f$ uniformly on A we can choose $m \geq m_1$ so that $n \geq m \Longrightarrow \big|f_n(x) - f(x)\big| < \delta$ for all $x \in A$. Let $n \geq m$ and let $x \in A$. Then we have $f_n(x), f(x) \in \big[-(M+1), M+1\big]$ with $\big|f_n(x) - f(x)\big| < \delta$ and hence $\big|g\big(f_n(x)\big) - g\big(f(x)\big)\big| < \epsilon$.

3: (a) Approximate $2^{-1/5}$ by a rational number so that the error is at most $\frac{1}{40}$.

Solution: By Theorem 4.40 (the sum of the binomial series)

$$\begin{aligned} 2^{-1/5} &= \left(1-\tfrac{1}{2}\right)^{1/5} = \sum_{n=0}^{\infty} \binom{1/5}{n}\left(-\tfrac{1}{2}\right)^n = 1 - \left(\tfrac{1}{5}\right)\left(\tfrac{1}{2}\right) - \tfrac{\left(\frac{1}{5}\right)\left(\frac{4}{5}\right)}{2!}\left(\tfrac{1}{2}\right)^2 - \tfrac{\left(\frac{1}{5}\right)\left(\frac{4}{5}\right)\left(\frac{9}{5}\right)}{3!}\left(\tfrac{1}{2}\right)^3 - \cdots \\ &= 1 - \tfrac{1}{10} - \tfrac{1\cdot 4}{10^2 2!} - \tfrac{1\cdot 4\cdot 9}{10^3 3!} - \tfrac{1\cdot 4\cdot 9\cdot 14}{10^4 4!} - \cdots \\ &\cong 1 - \tfrac{1}{10} - \tfrac{1\cdot 4}{10^2 2!} = 1 - \tfrac{1}{10} - \tfrac{1}{50} = \tfrac{22}{25} \end{aligned}$$

with error

$$\begin{aligned} E &= \tfrac{1\cdot 4\cdot 9}{10^3 3!} + \tfrac{1\cdot 4\cdot 9\cdot 14}{10^4 4!} + \tfrac{1\cdot 4\cdot 9\cdot 14\cdot 19}{10^5 5!} + \cdots \ = \tfrac{1\cdot 4\cdot 9}{10^3 3!}\left(1 + \tfrac{14}{10\cdot 4} + \tfrac{14\cdot 19}{10^2\cdot 4\cdot 5} + \tfrac{14\cdot 19\cdot 24}{10^3\cdot 4\cdot 5\cdot 6} + \cdots\right) \\ &< \tfrac{1\cdot 4\cdot 9}{10^3 3!}\left(1 + \tfrac{20}{10\cdot 4} + \tfrac{20\cdot 25}{10^2\cdot 4\cdot 5} + \tfrac{20\cdot 25\cdot 30}{10^3\cdot 4\cdot 5\cdot 6} + \cdots\right) = \tfrac{6}{1000}\left(1 + \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^3 + \cdots\right) = \tfrac{12}{1000} < \tfrac{25}{1000} = \tfrac{1}{40}\,. \end{aligned}$$

(b) Evaluate $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$.

Solution: Let $S = \sum_{n=1}^{\infty} \frac{n^3}{3^n}$ For $|x| < 1$ we have $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. Differentiate to get $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n\,x^{n-1}$. Multiply by x to get $\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n\,x^n$. Differentiate again to get $\frac{1+x}{(1-x)^3} = \sum_{n=1}^{\infty} n^2 x^{n-1}$. Multiply by x again to get $\frac{x+x^2}{(1-x)^3} = \sum_{n=1}^{\infty} n^2 x^n$. Differentiate a third time to get $\frac{1+4x+x^2}{(1-x)^4} = \sum_{n=1}^{\infty} n^3 x^{n-1}$. Finally, multiply by x to get $\frac{x+4x^2+x^3}{(1-x)^4} = \sum_{n=1}^{\infty} n^3 x^n$. Put in $x = \frac{1}{3}$ to get $S = \frac{\frac{1}{3}+\frac{4}{9}+\frac{1}{27}}{\left(\frac{2}{3}\right)^4} = \frac{33}{8}$.

(c) Evaluate $\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}\binom{2n}{n}$.

Solution: Let $a_n = \frac{(-1)^n}{4^n}\binom{2n}{n}$. For $n \geq 1$ we have

$$|a_n| = \tfrac{1}{4^n}\binom{2n}{n} = \tfrac{(2n)!}{(2^n n!)^2} = \tfrac{1\cdot 2\cdot 3\cdot\ldots\cdot 2n}{(2\cdot 4\cdot 6\cdot\ldots\cdot 2n)^2} = \tfrac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\ldots\cdot 2n}.$$

Since $|a_n| = \frac{3}{2}\cdot\frac{5}{4}\cdot\ldots\cdot\frac{2n-1}{2n-2}\cdot\frac{1}{2n} \geq \frac{1}{2n}$ and $\sum_{n=1}^{\infty}\frac{1}{2n}$ diverges, it follows that $\sum |a_n|$ diverges by the Comparison Test. Since $a_0 = 1$ and $|a_n| = \frac{2n-1}{2n}|a_{n-1}| \leq |a_{n-1}|$ for $n \geq 1$, it follows that the sequence $\big(|a_n|\big)$ is decreasing. Since

$$|a_n|^2 = \tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}\cdot\tfrac{5}{6}\cdot\ldots\cdot\tfrac{2n-1}{2n}\cdot\tfrac{2n-1}{2n} \leq \tfrac{1}{2}\cdot\tfrac{2}{3}\cdot\tfrac{3}{4}\cdot\tfrac{4}{5}\cdot\ldots\cdot\tfrac{2n-1}{2n}\cdot\tfrac{2n}{2n+1} = \tfrac{1}{2n+1}$$

we have $|a_n| \leq \frac{1}{\sqrt{2n+1}} \longrightarrow 0$ as $n \to \infty$. Thus $\sum a_n = \sum (-1)^n |a_n|$ converges by the Alternating Series Test. Thus $\sum a_n$ is conditionally convergent. Note that

$$\tfrac{(-1)^n}{4^n}\binom{2n}{n} = \tfrac{(-1)^n}{4^n}\cdot\tfrac{(2n)!}{(n!)^2} = (-1)^n\,\tfrac{1\cdot 2\cdot 3\cdots(2n)}{(2\cdot 4\cdot 6\cdots(2n))^2} = (-1)^n\,\tfrac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} = \tfrac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots(-\frac{2n-1}{2})}{n!} = \binom{-1/2}{n}$$

so for $|x| < 1$, by Theorem 4.40 (the sum of the binomial series) we have

$$(1+x)^{-1/2} = \sum_{n=0}^{\infty}\binom{-1/2}{n}x^n = \sum_{n=0}^{n}\tfrac{(-1)^n}{4^n}\binom{2n}{n}x^n.$$

Since $\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}\binom{2n}{n}$ converges (conditionally), it follows from Abel's Theorem (Part 4 of Theorem 4.23) that $\sum_{n=0}^{n}\frac{(-1)^n}{4^n}\binom{2n}{n}x^n$ converges uniformly on $[0,1]$ and hence by Theorem 4.14 (uniform convergence and continuity) its sum $g(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}\binom{2n}{n}x^n$ is continuous on $[0,1]$. Since $f(x) = (1+x)^{-1/2}$ is also continuous on $[0,1]$ with $f(x) = g(x)$ when $x < 1$, we have $g(1) = f(1)$, that is

$$\sum_{n=0}^{\infty}\tfrac{(-1)^n}{4^n}\binom{2n}{n} = f(1) = (1+1)^{-1/2} = \tfrac{1}{\sqrt{2}}\,.$$

4: (a) Let $s_n = \sum_{k=0}^{n} a_k$ for $n \geq 0$. Show that if the power series $\sum_{n=0}^{\infty} a_n x^n$ has a positive radius of convergence, then so does the power series $\sum_{n=0}^{\infty} s_n x^n$.

Solution: Let R be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_x^n$, and suppose that $R > 0$. Recall that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for all $|x| < 1$. Let $S = \min\{R, 1\}$. By the Multiplication of Power Series Theorem, since $\sum a_n x^n$ and $\sum x^n$ both converge for all $|x| < S$, the series $\sum s_n x^n$ also converges for all $|x| < S$ with

$$\sum_{n=0}^{\infty} s_n x^n = \Big(\sum_{n=0}^{n} a_n x^n\Big)\Big(\sum_{n=0}^{\infty} x^n\Big) = \Big(\sum_{n=0}^{\infty} a_n x^n\Big) \cdot \frac{1}{1-x}.$$

(b) (The Riemann Zeta Function) Define $\zeta : (1, \infty) \to \mathbb{R}$ by $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$. Prove that ζ is differentiable on $(1, \infty)$. Hint: use the Weierstrass M-Test, together with convergence tests from first year calculus, to show that for all $r > 1$ the series $\sum \frac{1}{n^x}$ and $\sum \frac{-\ln n}{n^x}$ both converge uniformly on $[r, \infty)$, then apply The Uniform Convergence and Differentiation Theorem.

Solution: Note that $\sum_{n\geq 1} \frac{1}{n^x}$ converges (it is a p-series with $p = x > 1$) and so $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$ is well-defined.

Let $r > 1$. Let $f_n(x) = \frac{1}{n^x}$ and note that $f_n{}'(x) = \frac{-\ln n}{n^x}$. For all $x \geq r$ we have $\big|f_n(x)\big| = \frac{1}{n^x} \leq \frac{1}{n^r}$, and the series $\sum \frac{1}{n^r}$ converges (its a p-series with $p = r$) and so the series $\sum f_n(x)$ converges uniformly on $[r, \infty)$ by the Weierstrass M-Test. Also, for all $x \geq r$ we have $\big|f_n{}'(x)\big| = \frac{\ln .n}{n^x} \leq \frac{\ln n}{n^r}$. Choose p with $1 < p < r$ and let $q = r - p > 0$. Then $\frac{\ln n}{n^r} = \frac{\ln n}{n^q} \cdot \frac{1}{n^p}$. By l'Hôpital's Rule, we have $\lim\limits_{n\to\infty} \frac{\ln n}{n^q} = \lim\limits_{n\to\infty} \frac{n^{-1}}{q\, n^{q-1}} = \lim\limits_{n\to\infty} \frac{1}{q\, n^q} = 0$, so, for sufficiently large n, we have $\frac{\ln n}{n^q} \leq 1$ hence $\frac{\ln n}{n^q} \cdot \frac{1}{n^p} \leq \frac{1}{n^p}$. Since $\frac{\ln n}{n^r} \leq \frac{1}{n^p}$ for large values of n, and the series $\sum \frac{1}{n^p}$ converges (since $p > 1$), it follows that the series $\sum \frac{\ln n}{n^r}$ converges by the Comparison Test. Since $\big|f_n(x)\big| \leq \frac{\ln n}{n^r}$ for all $x \in [r, \infty)$ and $\sum \frac{\ln n}{n^r}$ converges, it follows that $\sum f_n{}'(x)$ converges uniformly on $[r, \infty)$ by the Weierstrass M-Test. Since $\sum f_n(x)$ and $\sum f_n{}'$ converge uniforly on $[r, \infty)$, they also converge uniformly on $[r, s]$ for any value of $s > r$. By Theorem 4.16, it follows that the function $\zeta(x) = \sum_{n=1}^{\infty} f_n(x)$ is differentiable on $[r, s]$ for any value of $s > r$. Since ζ is differentiable on $[r, s]$ for every $1 \leq r < s$, it follows that ζ is differentiable on $(1, \infty)$. Indeed, given $a > 1$ we can choose r and s with $1 < r < a < s$ and then, since ζ is differentiable on $[r, s]$, it is differentiable at a.