**1:** For each of the following sequences of functions $(f_n)_{n\geq 1}$, find the set $A$ of points $x \in \mathbb{R}$ for which the sequence of real numbers $\big(f_n(x)\big)_{n\geq 1}$ converges, find the pointwise limit $g(x) = \lim\limits_{n\to\infty} f_n(x)$ for all $x \in A$, and determine whether $f_n \to g$ uniformly in $A$.

(a) $f_n(x) = (\sin x)^n$

(b) $f_n(x) = x\, e^{-nx^2}$

(c) $f_n(x) = x^n - x^{2n}$

**2:** Let $(a_n)_{n\geq 1}$ be a sequence in $\mathbb{R}$, let $(f_n)_{n\geq 1}$ be a sequence of functions $f_n : A \subseteq \mathbb{R} \to \mathbb{R}$, let $g : A \subseteq \mathbb{R} \to \mathbb{R}$ and let $h : \mathbb{R} \to \mathbb{R}$.

(a) Suppose that $\sum_{n\geq 1} a_n$ converges and $\big|f_{n+1}(x) - f_n(x)\big| \leq a_n$ for all $n \geq 1$ and all $x \in A$. Show that $(f_n)_{n\geq 0}$ converges uniformly on $A$.

(b) Suppose that $f_n \to g$ uniformly on $A$ and $f_n(x) \geq 0$ for all $n \geq 1$ and all $x \in A$. Show that $\sqrt{f_n} \to \sqrt{g}$ uniformly on $A$.

(c) Suppose that $f_n \to g$ uniformly on $A$, $g$ is bounded, and $h$ is continuous. Prove that $h \circ f_n \to h \circ g$ uniformly on $A$.

**3:** (a) Approximate $2^{-1/5}$ by a rational number so that the error is at most $\frac{1}{40}$.

(b) Evaluate $\sum\limits_{n=1}^{\infty} \frac{n^3}{3^n}$.

(c) Evaluate $\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{4^n} \binom{2n}{n}$.

**4:** (a) Let $s_n = \sum_{k=0}^{n} a_k$ for $n \geq 0$. Show that if the power series $\sum_{n=0}^{\infty} a_n x^n$ has a positive radius of convergence, then so does the power series $\sum_{n=0}^{\infty} s_n x^n$.

(b) (The Riemann Zeta Function) Define $\zeta : (1,\infty) \to \mathbb{R}$ by $\zeta(x) = \sum\limits_{n=1}^{\infty} \frac{1}{n^x}$. Prove that $\zeta$ is differentiable on $(1,\infty)$. Hint: use the Weierstrass M-Test, together with convergence tests from first year calculus, to show that for all $r > 1$ the series $\sum \frac{1}{n^x}$ and $\sum \frac{-\ln n}{n^x}$ both converge uniformly on $[r,\infty)$, then apply The Uniform Convergence and Differentiation Theorem.