PMATH 333 Real Analysis, Solutions to Assignment 2

1: Let $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ be sequences in $\mathbb{R}$.

(a) Prove, from the definition of the limit, that if $x_n = \frac{\sqrt{4n+1}}{\sqrt{n}}$, then $\lim\limits_{n\to\infty} x_n = 2$.

Solution: Note that for all $n \in \mathbb{Z}^+$ we have

$$\begin{aligned} \big|x_n - 2\big| &= \left|\tfrac{\sqrt{4n+1}}{\sqrt{n}} - 2\right| = \left|\tfrac{\sqrt{4n+1}-\sqrt{4n}}{\sqrt{n}}\right| = \left|\tfrac{\sqrt{4n+1}-\sqrt{4n}}{\sqrt{n}} \cdot \tfrac{\sqrt{4n+1}+\sqrt{4n}}{\sqrt{4n+1}+\sqrt{4n}}\right| \\ &= \tfrac{1}{\sqrt{n}(\sqrt{4n+1}+\sqrt{4n})} < \tfrac{1}{\sqrt{n}(\sqrt{4n}+\sqrt{4n})} = \tfrac{1}{4n}. \end{aligned}$$

Given $\epsilon > 0$ we choose $m \in \mathbb{Z}^+$ such that $\frac{1}{4m} < \epsilon$. Then when $n \geq m$ we have $|x_n - 2| < \frac{1}{4n} \leq \frac{1}{4m} < \epsilon$.

(b) Prove that if $x_n \geq 0$ for all $n \geq 1$ and $\lim\limits_{n\to\infty} x_n = 0$ then the set $\{x_n \big| n \in \mathbb{Z}^+\}$ has a maximum element.

Solution: Suppose that $x_n \geq 0$ for all $n \geq 1$ and that $\lim\limits_{n\to\infty} x_n = 0$ and let $S = \{x_n | n \in \mathbb{Z}^+\}$. If $x_n = 0$ for all $n \geq 1$ then $S = \{0\}$ and $b = 0$ is the maximum element of S. Suppose that there is at least one value of n such that $x_n > 0$. Choose $\ell \geq 1$ such that $x_\ell > 0$. Since $\lim\limits_{n\to\infty} x_n = 0$ we can choose $m \geq 1$ so that $n \geq m \Longrightarrow x_n < x_\ell$ (we remark that if we had $\ell \geq m$ then we would have $x_\ell < x_\ell$ which is impossible, so we must have $m > \ell$). Let $b = \max\{x_1, x_2, \cdots, x_\ell, \cdots, x_m\}$. Then $b = x_k$ for some $1 \leq k \leq m$, so we have $b \in S$, and $x_n \leq x_\ell \leq b$ for all $n \geq 1$ so we have $b = \max(S)$.

(c) Prove that if $\lim\limits_{n\to\infty} x_n = a > 0$ and $y_n > 0$ for all $n \geq 1$ with $\lim\limits_{n\to\infty} y_n = 0$ then $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = \infty$.

Solution: Suppose that $\lim\limits_{n\to\infty} x_n = a > 0$ and $y_n > 0$ for all $n \geq 1$ with $\lim\limits_{n\to\infty} y_n = 0$. Let $R > 0$. Choose a real number r with $0 < r < a$. Since $\lim\limits_{n\to\infty} x_n = a > r$ we can choose $m_1 \geq 1$ so that $n \geq m_1 \Longrightarrow x_n > r$. Since $y_n > 0$ and $\lim\limits_{n\to\infty} y_n = 0$ we can choose $m_2 \geq 1$ so that $n \geq m_2 \Longrightarrow 0 < y_n < \frac{r}{R}$. Let $m = \max\{m_1, m_2\}$. Then for $n \geq m$ we have $x_n > r$ and $0 < y_n < \frac{r}{R}$ and so $\frac{x_n}{y_n} > \frac{r}{r/R} = R$. Thus $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = \infty$.

(d) Prove that if $(x_n)_{n\geq 1}$ is increasing, and $(y_n)_{n\geq 1}$ converges, and we have $\big|x_n - y_n\big| < \frac{2n}{n+1}$ for all $n \in \mathbb{Z}^+$, then $(x_n)_{n\geq 1}$ converges.

Solution: Suppose that $(x_n)_{n\geq 1}$ is increasing and $(y_n)_{n\geq 1}$ converges and $\big|x_n - y_n\big| < \frac{2n}{n+1}$ for all $n \in \mathbb{Z}^+$. Let $b = \lim\limits_{n\to\infty} y_n$. Choose $m \geq 1$ so that $n \geq m \Longrightarrow |y_n - b| < 1 \Longrightarrow b - 1 < y_n < b + 1$. Then for all $n \geq m$ we have $|x_n - y_n| < \frac{2n}{n+1} < 2$ so that $y_n - 2 < x_n < y_n + 2$, and we have $b - 1 < y_n < b + 1$, and it follows that $x_n < y_n + 2 < b + 3$. Since $x_n < b + 3$ for all $n \geq m$, the sequence $(x_n)_{n\geq 1}$ is bounded above (by the maximum of $x_1, x_2, \cdots, x_{m-1}$ and $b+3$). Since the sequence $(x_n)_{n\geq 1}$ is increasing and bounded above, it converges by the Monotone Convergence Theorem.

2: We denote the set of *extended real numbers* by $[-\infty,\infty]$ (or by $\mathbb{R}\cup\{\pm\infty\}$). This is an ordered set with maximum element ∞ and minimum element $-\infty$. Note that every nonempty set $A\subseteq\mathbb{R}$ has a supremum and an infimum in $[-\infty,\infty]$ (when A is not bounded above in $\mathbb{R}$ we have $\sup A=\infty$, and when A is not bounded below in $\mathbb{R}$ we have $\inf A=-\infty$). For a sequence $(x_n)_{n\ge 1}$ in $\mathbb{R}$, we define the *limit supremum* and the *limit infimum* of $(x_n)_{n\ge1}$ to be the following extended real numbers:

$$\limsup_{n\to\infty} x_n=\lim_{n\to\infty}u_n,\text{ where } u_n=\sup\left\{x_k\,\middle|\,k\ge n\right\}\text{ , and }\liminf_{n\to\infty}x_n=\lim_{n\to\infty}\ell_n,\text{ where }\ell_n=\inf\left\{x_k\,\middle|\,k\ge n\right\}.$$

(a) Explain why $\limsup\limits_{n\to\infty}x_n$ and $\liminf\limits_{n\to\infty}x_n$ always exist in $[-\infty,\infty]$ for every sequence $(x_n)_{n\ge1}$ in $\mathbb{R}$.

Solution: Note that when $\emptyset\ne S\subseteq T\subseteq\mathbb{R}$, every upper bound of T (including ∞) is also an upper bound of S, and so we have $\sup T\ge\sup S$. Likewise, every lower bound of T is also a lower bound of S, and so $\inf T\le\inf S$. If we let $S_n=\{x_k\,|\,k\ge n\}$ then we have $S_1\supseteq S_2\supseteq S_3\supseteq\cdots$ and so $\sup S_1\ge\sup S_2\ge\sup S_3\ge\cdots$, that is $u_1\ge u_2\ge u_3\ge\cdots$ in $[-\infty,\infty]$. If $u_n=\infty$ for all n (that is, if every set S_n is not bounded above) then we shall agree that $\limsup\limits_{n\to\infty}x_n=\lim\limits_{n\to\infty}u_n=\infty$.

Suppose that $u_m<\infty$ for some m. Then S_m is bounded above, hence S_1 is bounded above (by the maximum of $x_1,x_2,\cdots,x_{m-1}$, and $\sup S_m$), and hence every S_n is bounded above (since $S_n\subseteq S_1$). We also note that $u_n\ge x_n>-\infty$ so that $(u_n)_{n\ge1}$ is a decreasing sequence of real numbers, and so it has a limit in the extended real numbers by the Monotone Convergence Theorem (if the sequence (u_n) is bounded below then it has a finite limit in $\mathbb{R}$, and if it is not bounded below then the limit is $-\infty$). Similarly, the sequence $(\ell_n)_{n\ge1}$ is an increasing sequence in $[-\infty,\infty)$, so it has a limit in the extended real numbers.

(b) Find $\limsup\limits_{n\to\infty}x_n$ and $\liminf\limits_{n\to\infty}x_n$ for the sequence given by $x_1=0$, $x_{2k}=\frac12\,x_{2k-1}$ and $x_{2k+1}=\frac12+x_{2k}$.

Solution: By induction, we have $x_{2k}=\frac12-\frac1{2^k}$ and $x_{2k+1}=1-\frac1{2^k}$ for all $k\ge1$. We claim that $\limsup\limits_{n\to\infty}x_n=1$ and $\liminf\limits_{n\to\infty}x_n=\frac12$. Let $S_n=\{x_k|k\ge n\}$. Since $x_{2j}=\frac12-\frac1{2^j}<\frac12<1$ for all j and $x_{2j+1}=1-\frac1{2^j}<1$ for all j, we have $x_k<1$ for all k, and so 1 is an upper bound for each of the sets S_n. Thus, for every $n\ge1$, we have $u_n=\sup S_n\le1$. On the other hand, we cannot have $u_n<1$ because given any number $r<1$ and any $n\ge1$ we can choose $k=2j+1\ge n$ so that $x_k=a_{2j+1}=1-\frac1{2^j}>r$ showing that r is not an upper bound for S_n. Thus $u_n=\sup S_n=1$ for all $n\ge1$, and so $\limsup\limits_{n\to\infty}x_n=\lim\limits_{n\to\infty}u_n=1$.

For $n=2m$, the set $S_n=S_{2m}$ contains the elements x_{2k} with $k\ge m$ and the elements x_{2k+1} with $k\ge m$. Note that $x_{2m}=\frac12-\frac1{2^m}$ is a lower bound for S_{2m} because when $k\ge m$ we have $\frac1{2^k}\ge\frac1{2^m}$ so $x_{2k}=\frac12-\frac1{2^k}\ge\frac12-\frac1{2^m}$ and $x_{2k+1}=1-\frac1{2^k}\ge1-\frac1{2^m}>\frac12-\frac1{2^m}$. Since x_{2m} is a lower bound for S_{2m} we have $\inf S_{2m}\ge x_{2m}$. On the other hand, since $x_{2m}\in S_{2m}$ we have $\inf S_{2m}\le x_{2m}$. It follows that $\ell_{2m}=\inf S_{2m}=x_{2m}=\frac12-\frac1{2^m}$ for all $m\ge1$. A fairly similar (but not identical) argument shows that $\ell_{2m-1}=\inf S_{2m-1}=x_{2m}$ for all $m\ge1$. Since $\ell_{2m}=\ell_{2m-1}=\frac12-\frac1{2^m}$ we have $\liminf\limits_{n\to\infty}x_n=\lim\limits_{n\to\infty}\ell_n=\frac12$.

(c) Show that for any sequence $(x_n)_{n\ge1}$ in $\mathbb{R}$, and for $c\in[-\infty,\infty]$, we have $\lim\limits_{n\to\infty}x_n=c$ if and only if $\limsup\limits_{n\to\infty}x_n=\liminf\limits_{n\to\infty}x_n=c$.

Solution: Let $u_n=\sup\{x_k|k\ge n\}$ and $\ell_n=\inf\{x_k|k\ge n\}$. Note that for all $n\ge1$ we have $\ell_n\le x_n\le u_n$, and it follows byy comparison that $\liminf\limits_{n\to\infty}x_n=\lim\limits_{n\to\infty}\ell_n\le\lim\limits_{n\to\infty}u_n=\limsup\limits_{n\to\infty}x_n$.

Suppose that $\lim\limits_{n\to\infty}x_n=c\in[-\infty,\infty]$. Consider the case $c=\infty$. Let $r>0$. Choose $m\ge1$ so that $k\ge m\implies x_k>r$. Then for all $n\ge m$, r is a lower bound for $\{x_k|k\ge n\}$, so we have $r\le\ell_n$. This shows that $\lim\limits_{n\to\infty}\ell_n=\infty$ and hence $\liminf\limits_{n\to\infty}x_n=\lim\limits_{n\to\infty}\ell_n=\infty$. Since $\liminf\limits_{n\to\infty}x_n\le\limsup\limits_{n\to\infty}x_n$, we also have $\limsup\limits_{n\to\infty}x_n=\infty$. Similarly, in the case $c=-\infty$ we have $\lim\limits_{n\to\infty}u_n=-\infty$, $\limsup\limits_{n\to\infty}x_n=-\infty$ and $\liminf\limits_{n\to\infty}x_n=-\infty$. Consider the case $c\in\mathbb{R}$. Let $\epsilon>0$. Choose $m\ge1$ so that $k\ge m\implies c-\epsilon<x_k<c+\epsilon$. For all $n\ge m$ we have $x_k<c+\epsilon$ for all $k\ge n$ and hence $u_n\le c+\epsilon$. It follows that $\lim\limits_{n\to\infty}u_n\le c+\epsilon$. Since $\epsilon>0$ was arbitrary, it follows that $\limsup\limits_{n\to\infty}x_n=\lim\limits_{n\to\infty}u_n\le c$. Similarly, we have $\liminf\limits_{n\to\infty}x_n=\lim\limits_{n\to\infty}\ell_n\ge c$. Since $c\le\liminf\limits_{n\to\infty}x_n\le\limsup\limits_{n\to\infty}x_n\le c$, we have $\liminf\limits_{n\to\infty}x_n=\limsup\limits_{n\to\infty}x_n=c$.

Suppose, conversely, that $\limsup\limits_{n\to\infty}x_n=\liminf\limits_{n\to\infty}x_n=c\in[-\infty,\infty]$. If $c=\infty$ then since $\ell_n\le x_n$ for all n and $\lim\limits_{n\to\infty}\ell_n=c=\infty$ we have $\lim\limits_{n\to\infty}x_n=\infty$ (by comparison). If $c=-\infty$ then since $x_n\le u_n$ for all n and $\lim\limits_{n\to\infty}u_n=c=-\infty$ we have $\lim\limits_{n\to\infty}x_n=-\infty$ (by comparison). If $c\in\mathbb{R}$ then since $\ell_n\le x_n\le u_n$ and $\lim\limits_{n\to\infty}\ell_n=c=\lim\limits_{n\to\infty}u_n$ we have $\lim\limits_{n\to\infty}x_n=c$ (by the Squeeze Theorem).

3: Let $m \in \mathbb{Z}$ with $m \geq 2$, and let $S_m = \{0, 1, 2, \cdots, m-1\}$. In this problem we explore the base m representation of a real number.

(a) Let $a_1, a_2, a_3, \cdots \in S_m$. For $n \in \mathbb{Z}^+$, let $s_n = \sum_{k=1}^{n} \frac{a_k}{m^k}$. Show that the sequence $(s_n)_{n\geq 1}$ converges and that its limit lies in $[0, 1]$.

Solution: Since $a_k \geq 0$ for all k, we have $s_n = \sum_{k=1}^{n} \frac{a_k}{m^k} \geq 0$ for all n. Since $a_k \leq m-1$ for all k we have

$$s_n = \sum_{k=1}^{n} \frac{a_k}{m^k} \leq \sum_{k=1}^{n} \frac{m-1}{m^n} = (m-1) \sum_{k=1}^{n} \frac{1}{m^k} = (m-1) \frac{1 - \frac{1}{m^n}}{m-1} = 1 - \frac{1}{m^n} \leq 1$$

for all n. Since $s_n - s_{n-1} = \frac{a_n}{m^n} \geq 0$ for all n, the sequence (s_n) is increasing. Since (s_n) is increasing and bounded above (by 1), it converges by the Monotone Convergence Theorem. Since $0 \leq s_n \leq 1$ for all n we have $0 \leq \lim_{n\to\infty} s_n \leq 1$ by the Comparison Theorem.

(b) Given $x \in [0, 1]$ show that there exist $a_1, a_2, a_3, \cdots \in S_m$ such that for $s_n = \sum_{k=1}^{n} \frac{a_k}{m^k}$ we have $x = \lim_{n\to\infty} s_n$.

Solution: Let $x \in [0, 1]$. If $x = 1$ then we can choose $a_k = m - 1$ for all k to get

$$s_n = \sum_{k=1}^{n} \frac{a_k}{m^k} = \sum_{k=1}^{n} \frac{m-1}{m^k} = (m-1) \sum_{k=1}^{n} \frac{1}{m^k} = (m-1) \frac{1 - \frac{1}{m^n}}{m-1} = 1 - \frac{1}{m^n} \longrightarrow 1 = x.$$

Suppose that $x \in [0, 1)$. Then we have $0 \leq mx < m$ so we can choose $a_1 \in \{0, 1, \cdots, m-1\}$ so that $a_1 \leq mx < a_1 + 1$ (to be explicit, we choose $a_1 = \lfloor mx \rfloor$) then we have $\frac{a_1}{m} \leq x < \frac{a_1}{m} + \frac{1}{m}$. Suppose that we have constructed $a_1, a_2, \cdots, a_{n-1}$ with each $a_k \in \{0, 1, \cdots, m-1\}$ such that for $s_{n-1} = \sum_{k=1}^{n-1} \frac{a_k}{m^k}$ we have $s_{n-1} \leq x < s_{n-1} + \frac{1}{m^{n-1}}$. Then we have $0 \leq x - s_{n-1} < \frac{1}{m^{n-1}}$ hence $0 \leq m^n(x - s_{n-1}) < m$. We choose $a_n \in \{0, 1, \cdots, m-1\}$ so that $a_n \leq m^n(x - s_{n-1}) < a_n + 1$ (to be explicit, we choose $a_n = \lfloor m^n(x - s_{n-1}) \rfloor$) and then we have $\frac{a_n}{m^n} \leq x - s_{n-1} < \frac{a_n+1}{m^n}$ and so $s_{n-1} + \frac{a_n}{m^n} \leq x < s_{n-1} + \frac{a_n}{m^n} + \frac{1}{m^n}$, that is $s_n \leq x < s_n + \frac{1}{m^n}$. In this way, we obtain a sequence $a_1, a_2, a_3, \cdots$ with each $a_k \in \{0, 1, \cdots, m-1\}$ such that $s_n \leq x < s_n + \frac{1}{m^n}$ for all $n \geq 0$ (where we set $s_0 = 0$). Since $(s_n)_{n\geq 0}$ is increasing and bounded above (by x), it converges. Since $s_n \leq x < s_n + \frac{1}{m^n}$ for all $n \geq 1$, the Comparison Theorem gives $\lim_{n\to\infty} s_n \leq x \leq \lim_{n\to\infty} s_n$, so that $\lim_{n\to\infty} s_n = x$, as required.

(c) Let $a_1, a_2, a_3, \cdots \in S_m$ and $b_1, b_2, b_3, \cdots \in S_m$. Let $s_n = \sum_{k=1}^{n} \frac{a_k}{m^k}$ and let $t_n = \sum_{k=1}^{n} \frac{b_k}{m^k}$. Suppose there exists $p \in \mathbb{Z}$ with $p \geq 1$ such that $a_k = b_k$ for all $k < p$, $a_p = b_p + 1$, $a_k = 0$ for all $k > p$ and $b_k = m - 1$ for all $k > p$. Show that $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n$.

Solution: Let $x = \lim_{n\to\infty} s_n$. Choose p as above. Then for $n \geq p$ we have $s_n = \sum_{k=1}^{p-1} \frac{a_k}{m^k} + \frac{b_p+1}{m^p}$ which is constant (independent of n) and so $x = \lim_{n\to\infty} s_n = \sum_{k=1}^{p-1} \frac{a_k}{m^k} + \frac{b_p+1}{m^p}$. Also, for $n \geq p$ we have

$$\begin{aligned} t_n &= \sum_{k=1}^{p-1} \frac{b_k}{m^k} + \frac{b_p}{m^p} + \sum_{k=p+1}^{n} \frac{m-1}{m^k} = \sum_{k=1}^{p-1} \frac{a_k}{m^k} + \frac{b_p}{m^p} + (m-1) \sum_{k=p+1}^{n} \frac{1}{m^k} \\ &= \sum_{k=1}^{p-1} \frac{a_k}{m^k} + \frac{b_p}{m^p} + (m-1) \frac{\frac{1}{m^p} - \frac{1}{m^n}}{m-1} = \sum_{k=1}^{p-1} \frac{a_k}{m^k} + \frac{b_p}{m^p} + \frac{1}{m^p} - \frac{1}{m^n} \\ &= x - \frac{1}{m^n} \longrightarrow x \quad \text{as } n \to \infty. \end{aligned}$$

4: (a) Define $f,g:\mathbb{R}\to\mathbb{R}$ by $f(x)=x^3$ and $g(x)=\sqrt[3]{x}$. Show that g is uniformly continuous but that f is not.

Solution: We claim that $f(x)$ is not uniformly continuous. Choose $\epsilon=1$. Let $\delta>0$ Choose $a=\frac{1}{\delta}$ and $x=\delta+\frac{1}{\delta}$. Then $|x-a|=\delta$ and we have

$$\left|f(x)-f(a)\right|=\left(\delta+\tfrac{1}{\delta}\right)^3-\left(\tfrac{1}{\delta}\right)^3=3\delta+3\cdot\tfrac{1}{\delta}+\delta^3>3\left(\delta+\tfrac{1}{\delta}\right)>3>\epsilon$$

because when $\delta\geq 1$ we have $\delta+\frac{1}{\delta}>\delta\geq 1$ and when $0<\delta\leq 1$ we have $\delta+\frac{1}{\delta}>\frac{1}{\delta}\geq 1$. Thus f is not uniformly continuous.

We claim that g is uniformly continuous. First we note that for $\delta>0$ and for $a,x\in\mathbb{R}$, in the case that $|a|\leq 2\delta$, when $|x-a|<\delta$ we have $|x|<3\delta$ and so

$$|f(x)-f(a)|\leq|f(x)|+|f(a)|<(2\delta)^{1/3}+(3\delta)^{1/3}=(2^{1/3}+3^{1/3})\,\delta^{1/3}<3\,\delta^{1/3}$$

(because $3<\frac{27}{8}=\left(\frac{3}{2}\right)^3$ so that $3^{1/3}<\frac{3}{2}$ and hence $2^{1/3}+3^{1/3}<2\cdot\frac{3}{2}=3$) and in the case that $|a|\geq 2\delta$, when $|x-a|<\delta$, the numbers a and x have the same sign and we have $|x|\geq\delta$ and so

$$\begin{aligned}|f(x)-f(a)|=|x^{1/3}-a^{1/3}|&=\left|\frac{x-a}{x^{2/3}+x^{1/3}a^{1/3}+a^{2/3}}\right|=\frac{|x-a|}{|x|^{2/3}+|x|^{1/3}|a|^{1/3}+|a|^{2/3}}\\&<\frac{\delta}{\delta^{2/3}+\delta^{1/3}(2\delta)^{1/3}+(2\delta)^{2/3}}=\frac{\delta^{1/3}}{1+2^{1/3}+4^{1/3}}<\delta^{1/3}<3\,\delta^{1/3}.\end{aligned}$$

Thus given $\epsilon>0$ we can choose $\delta=\frac{1}{27}\,\epsilon^3$ so that $3\,\delta^{1/3}=\epsilon$ and then for all $a,x\in\mathbb{R}$ with $|x-a|<\delta$ we have $|f(x)-f(a)|<3\,\delta^{1/3}=\epsilon$. Thus g is uniformly continuous.

(b) Define $f:[0,1)\to\mathbb{R}$ as follows. Given $x\in[0,1)$, write x in its binary (base 2) representation as $x=[.a_1a_2a_3\cdots]_2=\sum_{k=1}^\infty\frac{a_k}{2^k}$ with each $a_k\in\{0,1\}$ so that $\forall m\in\mathbb{Z}^+\ \exists k\geq m\ a_k\neq 1$, then let $f(x)$ be the number whose ternary (base 3) representation is $f(x)=[.a_1a_2a_3\cdots]_3=\sum_{k=1}^\infty\frac{a_k}{3^k}$. Determine where the function f is continuous.

Solution: We claim that f is continuous from the right at all points $a\in[0,1)$ and f is continuous from the left at all points $a\in[0,1)$ except for the points of the form $a=\frac{k}{2^n}$ where $k\in\mathbb{Z}^+$ with $0<k<2^n$, in other words, except for the points $0\neq a\in[0,1)$ with finite base 2 representations.

First, let us show that f is continuous from the right at all points $a\in[0,1)$. Let $a\in[0,1)$. Write a in base 2 as $a=[.a_1a_2a_3\cdots]_2$ where $\forall m\in\mathbb{Z}^+\ \exists k\geq m\ \ a_k=0$. Let $\epsilon>0$. Choose $m\in\mathbb{Z}^+$ with $2\cdot 3^{-m}<\epsilon$, choose $k\geq m$ such that $a_k=0$, and let $\delta=2^{-k}$. For $x\in[0,1)$ with $a\leq x<a+\delta$, we shall prove on the next page that the base 2 representations of a and x are of the form $a=[.a_1a_2\cdots a_{k-1}0\,a_{k+1}a_{k+2}\cdots]_2$ and $x=[.a_1a_2\cdots a_{k-1}b_kb_{k+1}\cdots]_2$ with $b_k\in\{0,1\}$ and with $[.0\cdots 0\,b_kb_{k+1}b_{k+2}\cdots]_2\geq[.0\cdots 0\,a_{k+1}a_{k+2}\cdots]_2$. Note that in base 3 we also have $[.0\cdots 0\,b_kb_{k+1}\cdots]_3\geq[.0\cdots 0\,a_{k+1}a_{k+2}\cdots]_3$ and so

$$\begin{aligned}\left|f(x)-f(a)\right|=f(x)-f(a)&=[.a_1a_2\cdots a_{k-1}b_kb_{k+1}\cdots]_3-[.a_1a_2\cdots a_{k-1}0\,a_{k+1}\cdots]_3\\&=[.0\cdots 0\,b_kb_{k+1}\cdots]_3-[.0\cdots 0\,a_{k+1}a_{k+2}\cdots]_3\\&\leq[.0\cdots 0\,b_kb_{k+1}\cdots]_3\leq 2\cdot 3^{-k}\leq 2\cdot 3^{-m}<\epsilon.\end{aligned}$$

Thus f is continuous from the right at a, as claimed.

A similar argument shows that when $a\in(0,1)$ does not have a finite base 2 representation, the map f is continuous from the left at a. Note that for such $a\in(0,1)$, its base 2 representation $[.a_1a_2\cdots]_2$ is such that $\forall m\in\mathbb{Z}^+\ \exists k\geq m\ a_k=1$. Given $\epsilon>0$ we choose $m\in\mathbb{Z}^+$ so that $2\cdot 3^{-m}<\epsilon$, then we choose $k\geq m$ so that $a_k=1$, and we take $\delta=2^{-k}$. For $a-\delta<x\leq a$ the base 2 representations of a and x are of the form $a=[.a_1a_2\cdots a_{k-1}1\,a_{k+1}\cdots]_2$ and $x=[.a_1a_2\cdots a_{k-1}b_kb_{k+1}\cdots]_2$ with $b_k\in\{0,1\}$ and with $[.0\cdots 0\,b_kb_{k+1}\cdots]_2\leq[.0\cdots 0\,1\,a_{k+1}\cdots]_2$. As above, we have $|f(x)-f(a)|=f(a)-f(x)\leq 2\cdot 3^{-k}<\epsilon$.

Finally, suppose that $a\in(0,1)$ has a finite base 2 representation, say $a=[.a_1a_2\cdots a_m]_2$ with $a_m=1$. We claim that f is not continuous from the left at the point a (that is $\exists\epsilon>0\ \forall\delta>0\ \exists x\in[0,1)$ with $|x-a|\leq\delta$ and $|f(x)-f(a)|>\epsilon$). Choose $\epsilon=3^{-m-1}$. Let $\delta>0$. Choose $k\in\mathbb{Z}^+$ with $k>m$ and $2^{-k}<\delta$. Choose $x=a-2^{-k}=[.a_1a_2\cdots a_{m-1}1]_2-[.0\cdots 01]_2=[.a_1a_2\cdots a_{m-1}011\cdots 1]_2$ where the final 1 is in position k. Then we have $|x-a|=2^{-k}<\delta$ but

$$|f(x)-f(a)|=f(a)-f(x)=[.a_1a_2\cdots a_{m-1}1]_3-[.a_1\cdots a_{m-1}011\cdots 1]_3=[.0\cdots 01\cdots 112]_3$$

where the first 1 is in position $m+1$ and the final 2 is in position k, and so $|f(x)-f(a)|>3^{-m-1}=\epsilon$.

Let $a = [.a_1 a_2 \cdots]_2$ with $a_k = 0$, and let $x = [.b_1 b_2 \cdots]_2$ where $\forall\, m \in \mathbb{Z}^+ \ \exists\, j \geq m \ b_j = 0$. Suppose that $a \leq x < a + \frac{1}{2^k}$. Here is a proof that $b_j = a_j$ for all $j < k$. Suppose that this is not true, and let ℓ be the smallest integer with $1 \leq \ell < k$ such that $a_\ell \neq b_\ell$. Case 1: suppose that $a_\ell = 1$ and $b_\ell = 0$. Since each $a_j, b_j \in \{0, 1\}$ so that $a_j - b_j \geq -1$, and since $b_j = 0$ for some $j > \ell$, we have

$$a - x = \tfrac{1}{2^\ell} + \sum_{j=\ell+1}^{\infty} \tfrac{a_j - b_j}{2^j} > \tfrac{1}{2^\ell} - \sum_{j=\ell+1}^{\infty} \tfrac{1}{2^j} = \tfrac{1}{2^\ell} - \tfrac{1}{2^\ell} = 0$$

which contradicts the fact that $x \geq a$. Case 2: suppose that $a_\ell = 0$ and $b_\ell = 1$. Then since $a_j, b_j \in \{0, 1\}$ so $b_j - a_j \geq -1$, and $a_k = 0$, we have

$$x - a = \tfrac{1}{2^\ell} + \sum_{j=\ell+1}^{k-1} \tfrac{b_j - a_j}{2^j} + \tfrac{b_k}{2^k} + \sum_{j=k+1}^{\infty} \tfrac{b_j - a_j}{2^j} \geq \tfrac{1}{2^\ell} - \sum_{j=\ell+1}^{k-1} \tfrac{1}{2^j} + \tfrac{0}{2^k} - \sum_{j=k+1}^{\infty} \tfrac{1}{2^j} = \tfrac{1}{2^{k-1}} - \tfrac{1}{2^k} = \tfrac{1}{2^k}$$

which contradicts the fact that $x < a + \frac{1}{2^k}$.