PMATH 333 Real Analysis, Assignment 2 Due: Fri Feb 16, 11:00 pm

1: Let $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ be sequences in $\mathbb{R}$.

(a) Prove, from the definition of the limit, that if $x_n = \frac{\sqrt{4n+1}}{\sqrt{n}}$, then $\lim\limits_{n\to\infty} x_n = 2$.

(b) Prove that if $x_n \geq 0$ for all $n \geq 1$ and $\lim\limits_{n\to\infty} x_n = 0$ then the set $\{x_n | n \in \mathbb{Z}^+\}$ has a maximum element.

(c) Prove that if $\lim\limits_{n\to\infty} x_n = a > 0$ and $y_n > 0$ for all $n \geq 1$ with $\lim\limits_{n\to\infty} y_n = 0$ then $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = \infty$.

(d) Prove that if $(x_n)_{n\geq 1}$ is increasing, and $(y_n)_{n\geq 1}$ converges, and we have $|x_n - y_n| < \frac{2n}{n+1}$ for all $n \in \mathbb{Z}^+$, then $(x_n)_{n\geq 1}$ converges.

2: We denote the set of *extended real numbers* by $[-\infty,\infty]$ (or by $\mathbb{R}\cup\{\pm\infty\}$). This is an ordered set with maximum element ∞ and minimum element $-\infty$. Note that every nonempty set $A \subseteq \mathbb{R}$ has a supremum and an infimum in $[-\infty,\infty]$ (when A is not bounded above in $\mathbb{R}$ we have $\sup A = \infty$, and when A is not bounded below in $\mathbb{R}$ we have $\inf A = -\infty$). For a sequence $(x_n)_{n\geq 1}$ in $\mathbb{R}$, we define the *limit supremum* and the *limit infimum* of $(x_n)_{n\geq 1}$ to be the following extended real numbers:

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} u_n, \text{ where } u_n = \sup\{x_k \mid k \geq n\}, \text{ and } \liminf_{n\to\infty} x_n = \lim_{n\to\infty} \ell_n, \text{ where } \ell_n = \inf\{x_k \mid k \geq n\}.$$

(a) Explain why $\limsup\limits_{n\to\infty} x_n$ and $\liminf\limits_{n\to\infty} x_n$ always exist in $[-\infty,\infty]$ for every sequence $(x_n)_{n\geq 1}$ in $\mathbb{R}$.

(b) Find $\limsup\limits_{n\to\infty} x_n$ and $\liminf\limits_{n\to\infty} x_n$ for the sequence given by $x_1 = 0$, $x_{2k} = \frac{1}{2}\, x_{2k-1}$ and $x_{2k+1} = \frac{1}{2} + x_{2k}$.

(c) Show that for any sequence $(x_n)_{n\geq 1}$ in $\mathbb{R}$, and for $c \in [-\infty,\infty]$, we have $\lim\limits_{n\to\infty} x_n = c$ if and only if $\limsup\limits_{n\to\infty} x_n = \liminf\limits_{n\to\infty} x_n = c$.

3: Let $m \in \mathbb{Z}$ with $m \geq 2$, and let $S_m = \{0,1,2,\cdots,m-1\}$. In this problem we explore the base m representation of a real number.

(a) Let $a_1, a_2, a_3, \cdots \in S_m$. For $n \in \mathbb{Z}^+$, let $s_n = \sum\limits_{k=1}^{n} \frac{a_k}{m^k}$. Show that the sequence $(s_n)_{n\geq 1}$ converges and that its limit lies in $[0,1]$.

(b) Given $x \in [0,1]$ show that there exist $a_1, a_2, a_3, \cdots \in S_m$ such that for $s_n = \sum\limits_{k=1}^{n} \frac{a_k}{m^k}$ we have $x = \lim\limits_{n\to\infty} s_n$.

(c) Let $a_1, a_2, a_3, \cdots \in S_m$ and $b_1, b_2, b_3, \cdots \in S_m$. Let $s_n = \sum\limits_{k=1}^{n} \frac{a_k}{m^k}$ and let $t_n = \sum\limits_{k=1}^{n} \frac{b_k}{m^k}$. Suppose there exists $p \in \mathbb{Z}$ with $p \geq 1$ such that $a_k = b_k$ for all $k < p$, $a_p = b_p + 1$, $a_k = 0$ for all $k > p$ and $b_k = m-1$ for all $k > p$. Show that $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} t_n$.

4: (a) Define $f, g : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$. Show that g is uniformly continuous but that f is not.

(b) Define $f : [0,1) \to \mathbb{R}$ as follows. Given $x \in [0,1)$, write x in its binary (base 2) representation as $x = [.a_1a_2a_3\cdots]_2 = \sum_{k=1}^{\infty} \frac{a_k}{2^k}$ with each $a_k \in \{0,1\}$ so that $\forall m \in \mathbb{Z}^+\ \exists k \geq m\ a_k \neq 1$, then let $f(x)$ be the number whose ternary (base 3) representation is $f(x) = [.a_1a_2a_3\cdots]_3 = \sum_{k=1}^{\infty} \frac{a_k}{3^k}$. Determine where the function f is continuous.