PMATH 333 Real Analysis, Solutions to Assignment 1

1: In this problem, you will use the rules R1-R9 which define rings and fields, exactly as stated in the lecture notes, along with the rule R0 which states that for every $a \in R$ we have $a \cdot 0 = 0$ and $0 \cdot a = 0$ (this is Part 4 of Properties of Rings). In your solution, your proof should be very detailed, using only one rule at each step in the proof, and explicitly indicating which rule is used at each step. Note, for example, that to prove that $0 + a = a$ you need to use both R3 and R2.

(a) Let R be a ring. Using only the rules R1-R7 which define a ring, together with rule R0 as stated above, prove that for all $a, b, c, d \in R$, if $a + c = 0$ and $ab + d = 0$ then $cb = d$ (this is Part 6 of Properties of Rings).

Solution: Let $a, b, c, d \in R$. Suppose that $a + c = 0$ and $ab + d = 0$. By R4, we can choose $e \in R$ such that $ab + e = 0$. Then

$$\begin{aligned}
ab + cb &= (a + c)b \text{ , by R7} \\
&= 0 \cdot b \text{ , since } a + c = 0 \\
&= 0 \text{ , by R0} \\
&= ab + d \text{ , since } ab + d = 0 \\
cb + ab &= d + ab \text{ , by R2} \\
(cb + ab) + e &= (d + ab) + e \text{ , since } cb + ab = d + ab \\
cb + (ab + e) &= d + (ab + e) \text{ , by R1} \\
cb + 0 &= d + 0 \text{ , since } ab + e = 0 \\
cb &= d \text{ , by R0.}
\end{aligned}$$

(b) Let F be a field. Using only the rules R1-R9 which define a field, together with rule R0 as stated above, prove that for all $x, y \in F$, if $x \cdot x = y \cdot y$ then either $x = y$ or $x + y = 0$.

Solution: Let $x, y \in F$. Suppose that $x \cdot x = y \cdot y$. Suppose $x + y \neq 0$. Since $x + y \neq 0$, by R9 we can choose $z \in F$ such that $(x + y) \cdot z = 1$. Then

$$\begin{aligned}
x &= x \cdot 1 \text{ , by R6} \\
&= x \cdot ((x + y) \cdot z) \text{ , since } (x + y) \cdot z = 1 \\
&= (x \cdot (x + y)) \cdot z \text{ , by R5} \\
&= (x \cdot x + x \cdot y) \cdot z \text{ , by R7} \\
&= (y \cdot y + x \cdot y) \cdot z \text{ , since } x \cdot x = y \cdot y \\
&= ((y + x) \cdot y) \cdot z \text{ , by R7} \\
&= ((x + y) \cdot y) \cdot z \text{ , by R1} \\
&= (y \cdot (x + y)) \cdot z \text{ , by R8} \\
&= y \cdot ((x + y) \cdot z) \text{ , by R5} \\
&= y \cdot 1 \text{ , since } (x + y) \cdot z = 1 \\
&= y \text{ , by R3.}
\end{aligned}$$

2: In this problem, you will use rules R1-R9 and O1-O5 which define an ordered field, exactly as stated in the lecture notes, along with rule R0, as stated in Problem 1. Your proof should be very detailed, using only one rule at each step in the proof, and explicitly indicating which rule is used at each step.

(a) Let F be an ordered field. Using only the rules R1-R9 and O1-O5 which define an ordered field, together with the rule R0 from Problem 1, prove that for all $x \in F$, if $0 \leq x$ and $x \leq 1$ then $x \cdot x \leq x$.

Solution: Let $x \in F$. Suppose that $0 \leq x$ and $x \leq 1$. Using R4, choose $u \in F$ such that $1 + u = 0$ (so that we have $u = -1$). Using R4, choose $y \in F$ such that $x + y = 0$ (so that we have $y = -x$). Then

$$\begin{aligned}
x + y &\leq 1 + y \text{ , by O4, since } x \leq 1 \\
0 &\leq 1 + y \text{ , since } x + y = 0 \\
0 &\leq x \cdot (1 + y) \text{ , by O5, since } 0 \leq x \text{ and } 0 \leq 1 + y \\
0 &\leq x \cdot 1 + x \cdot y \text{ , by R7} \\
0 &\leq x + x \cdot y \text{ , by R6} \\
0 + x \cdot x &\leq (x + x \cdot y) + x \cdot x \text{ , by O4} \\
x \cdot x + 0 &\leq (x + x \cdot y) + x \cdot x \text{ , by R2} \\
x \cdot x &\leq (x + x \cdot y) + x \cdot x \text{ , by R3} \\
x \cdot x &\leq x + (x \cdot y + x \cdot x) \text{ , by R1} \\
x \cdot x &\leq x + x \cdot (y + x) \text{ , by R7} \\
x \cdot x &\leq x + x \cdot (x + y) \text{ , by R2} \\
x \cdot x &\leq x + x \cdot 0 \text{ , since } x + y = 0 \\
x \cdot x &\leq x + 0 \text{ , by R0} \\
x \cdot x &\leq x \text{ , by R3}
\end{aligned}$$

(b) Let F be an ordered field. Using only the rules R1-R9 and O1-O5 which define an ordered field, together with the rule R0 from Problem 1, prove that for all $x, y \in F$, if $x \leq 0$ and $y \leq 0$ then $0 \leq xy$.

Solution: Let $x, y \in F$. Suppose that $0 \leq x$ and $0 \leq y$. Using R4, choose $u, v \in F$ such that $x + u = 0$ and $y + v = 0$. Note that

$$\begin{aligned}
uv &= uv + 0 \cdot y \text{ , by R3} \\
&= uv + 0 \cdot y \text{ , by R0} \\
&= uv + (x + u) \cdot y \text{ , since } x + u = 0 \\
&= uv + (xy + uy) \text{ , by R7} \\
&= (xy + uy) + uv \text{ , by R2} \\
&= xy + (uy + uv) \text{ , by R1} \\
&= xy + u(y + v) \text{ , by R7} \\
&= xy + u \cdot 0 \text{ , since } y + v = 0 \\
&= xy + 0 \text{ , by R0} \\
&= xy \text{ , by R3.}
\end{aligned}$$

Since $x \leq 0$ we have

$$\begin{aligned}
x + u &\leq 0 + u \text{ , by O4} \\
0 &\leq 0 + u \text{ , since } x + u = 0 \\
0 &\leq u + 0 \text{ , by R2} \\
0 &\leq u \text{ , by R3.}
\end{aligned}$$

Since $y \leq 0$ we have

$$\begin{aligned}
y + v &\leq 0 + v \text{ , by O4} \\
0 &\leq 0 + v \text{ , since } y + v = 0 \\
0 &\leq v + 0 \text{ , by R2} \\
0 &\leq v \text{ , by R3.}
\end{aligned}$$

Since $0 \leq u$ and $0 \leq v$ we have $0 \leq uv$ by O5, and since $uv = xy$, this gives $0 \leq xy$, as required.

3: In this problem, you can freely use any of the familiar properties which hold in ordered fields (you do not need to explicitly indicate when your proof uses these rules). But your must explicitly indicate each time your proof uses one of the named order properties involving $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$. To be specific, you must clearly indicate each time your proof uses any of the following properties: the Discreteness Property of $\mathbb{Z}$, the Least Upper (or Greatest Lower) Bound Property of $\mathbb{R}$, the Approximation Property of the Supremum or Infimum, the Well-Ordering Property of $\mathbb{Z}$ in $\mathbb{R}$, the Floor and Ceiling Properties of $\mathbb{Z}$ in $\mathbb{R}$, the Archimedian Property of $\mathbb{Z}$ in $\mathbb{R}$, the Density of $\mathbb{Q}$ in $\mathbb{R}$, the Induction Principle in $\mathbb{Z}$, or the Strong Induction Principle in $\mathbb{Z}$.

(a) Let $S = \left\{\frac{2+(-1)^n}{n} \middle| n \in \mathbb{Z}^+\right\}$. Find (with proof) $\sup S$ and $\inf S$.

Solution: Let $a_n = \frac{2+(-1)^n}{n}$ so that $S = \{a_n | n \in \mathbb{Z}^+\}$. We claim that $\sup S = \max S = \frac{3}{2}$. Note that $\frac{3}{2}$ is an upper bound for S because $a_1 = 1 < \frac{3}{2}$ and $a_2 = \frac{3}{2}$ and for $n \geq 2$ we have $a_n = \frac{2+(-1)^n}{n} \leq \frac{3}{n} \leq \frac{3}{3} = 1 < \frac{3}{2}$. Since $\frac{3}{2}$ is an upper bound for S with $\frac{3}{2} = a_2 \in S$ it follows that $\sup S = \max S = \frac{3}{2}$.

We claim that $\inf S = 0$. Note that 0 is a lower bound for S because $a_n = \frac{2+(-1)^n}{n} \geq \frac{1}{n} > 0$ for all $n \geq 1$. Let m be any lower bound for S. We need to show that $m \leq 0$. Suppose, for a contradiction, that $m > 0$. By the Archimedean Property, we can choose $n \in \mathbb{Z}^+$ with $n > \frac{3}{m}$ so that $\frac{3}{n} < m$. Then $a_n = \frac{2+(-1)^n}{n} \leq \frac{3}{n} < m$, which contradicts the fact that m is a lower bound for S. Thus $m \leq 0$, as required.

(b) Let A and B be nonempty bounded subsets of $\mathbb{R}$ and let $C = \left\{x+y \,\middle|\, x \in A, y \in B\right\}$. Prove that C is bounded and $\sup A + \sup B = \sup C$.

Solution: We claim that C is bounded. Let r and s be lower bounds for A and B, respectively, and let u and v be upper bounds for A and B. Let $z \in C$. Say $z = x + y$ with $x \in A$ and $y \in B$. Since $r \leq x \leq u$ and $s \leq y \leq v$ we have $r + s \leq x + y \leq u + v$. Thus $r + s \leq z$ for every $z \in C$, so $r + s$ is a lower bound for C, and $z \leq u + v$ for every $z \in C$, so $u + v$ is an upper bound for C. Thus C is bounded, as claimed.

Now let $u = \sup A$, $v = \sup B$ and $w = \sup C$ (these exist by the Least Upper Bound Property of $\mathbb{R}$). We need to show that $u + v = w$. Since u and v are upper bounds for A and B, it follows, as shown above, that $u+v$ is an upper bound for C, and so $w \leq u+v$ (since w is the least upper bound). It remains to show that $u + v \leq w$. Let $\epsilon > 0$. By the Approximation Property, we can choose $x \in A$ with $u - \frac{\epsilon}{2} < x$ and we can choose $y \in B$ with $v - \frac{\epsilon}{2} < y$. Then we have $x + y > \left(u - \frac{\epsilon}{2}\right) + \left(v - \frac{\epsilon}{2}\right) = (u + v) - \epsilon$. Since $x + y \in C$ and w is an upper bound for C we have $w \geq x + y > (u + v) - \epsilon$. Since $w > u + v - \epsilon$ for all $\epsilon > 0$ it follows that $w \geq u + v$.

(c) Let S be a nonempty set in $\mathbb{R}$ which is bounded above. Suppose there exists $\delta > 0$ such that for all $x, y \in S$, if $x \neq y$ then $|y - x| \geq \delta$. Prove that S has a maximum element.

Solution: Choose $\delta > 0$ so that $|y - x| \geq \delta$ for all $x, y \in S$. Since S is nonempty and bounded above, it has a supremum in $\mathbb{R}$ (by the Least Upper Bound Property). Let $b = \sup S$. We need to show that $b \in S$. Suppose, for a contradiction, that $b \notin S$. By the Approximation Property, we can choose $x \in S$ with $b - \delta < x \leq b$. Since $x \in S$ and $b \notin S$ we have $x \neq b$ and so $b - \delta < x < b$, hence also $b < x + \delta$. By the Approximation Property again, we can choose $y \in S$ with $x < y \leq b$. Since $y \in S$ and $b \notin S$ we have $y \neq b$ and so $x < y < b$. Thus $x < y < b < x + \delta$. But then $|y - x| = y - x < (x + \delta) - x = \delta$, and this contradicts the choice of δ.

4: Let $n \in \mathbb{Z}^+$ and let $0 < y \in \mathbb{R}$. In this problem you will prove that y has a unique positive n^{th} root $x \in \mathbb{R}$. You can freely use any of the rules and properties discussed in Chapter 1 (you do not need to explicitly indicate when the various rules and properties are being used).

(a) Show that for all $a, b \in \mathbb{R}$, if $0 < a < b$ then $b^n - a^n \leq n\, b^{n-1}(b-a)$.

Solution: When $n = 1$ we have $b^n - a^n = b - a$ and $nb^{n-1}(b-a) = b - a$, and when $n \geq 2$ we have

$$b^n - a^n = (b-a)(b^{n-1} + ab^{n-2} + \cdots + a^{n-2}b + a^{n-1}) < (b-a) \cdot nb^{n-1}.$$

(b) Let $A = \left\{0 < t \in \mathbb{R} \,\middle|\, t^n < y\right\}$. Show that A is nonempty and bounded above and let $x = \sup A$.

Solution: The set A is not empty because if $y > 1$ then we have $1^n = 1 < y$ so that $1 \in A$ and if $0 < y \leq 1$ then for $t = \frac{y}{2} < 1$ we have $t^n < t^{n-1} < \cdots < t^2 < t < y$ so that $t \in A$. The set A is bounded above because if $y \leq 1$ then A is bounded above by 1 since if $t > 1$ then $t^n > t > 1 > y$ so that $t \notin A$, and if $y \geq 1$ then A is bounded above by y since if $t > y \geq 1$ then $t^n > t > y$ so that $t \notin A$.

(c) Using Part (a), or otherwise, show that $x^n \geq y$, where x is as in Part (b).

Solution: Let $x = \sup A$, as above, and suppose, for a contradiction, that $x^n < y$. To obtain a contradiction, we wish to show that x is not an upper bound for A by finding $\epsilon > 0$ so that $x + \epsilon \in A$, that is so that $(x+\epsilon)^n < y$. To get $(x+\epsilon)^n < y$ we need $(x+\epsilon)^n - x^n < y - x^n$. From Part (a), we know that $(x+\epsilon)^n - x^n \leq \epsilon \cdot n(x+\epsilon)^{n-1}$. Choose $\epsilon > 0$ with $\epsilon \leq 1$ and $\epsilon < \frac{y - x^n}{n(x+1)^{n-1}}$. Then we have $(x+\epsilon)^n - x^n \leq \epsilon \cdot n(x+\epsilon)^{n-1} \leq \epsilon \cdot n(x+1)^{n-1} < y - x^n$. Thus $(x+\epsilon)^n < y$ and so $x + \epsilon \in A$ hence $x \neq \sup A$, giving the desired contradiction.

(d) Using Part (a), or otherwise, show that $x^n \leq y$.

Solution: Suppose, for a contradiction, that $x^n > y$. Choose $\epsilon > 0$ with $\epsilon < \frac{x^n - y}{nx^{n-1}}$. For all $t > 0$, if $t > x - \epsilon$ then $t^n > (x-\epsilon)^n$ so we have

$$x^n - t^n < x^n - (x-\epsilon)^n \leq \epsilon \cdot nx^{n-1} < x^n - y$$

and hence $t^n > y$ so that $t \notin A$. This shows that $x - \epsilon$ is an upper bound for A, which contradicts the fact that x is the least upper bound.

(e) Finally, show that if x_1 and x_2 are positive real numbers with ${x_1}^n = {x_2}^n$ then $x_1 = x_2$.

Solution: If $x_1, x_2 > 0$ with $x_1 \neq x_2$, say $0 < x_1 < x_2$, then we have $0 < {x_1}^n < {x_2}^n$ and hence $x_1^n \neq x_2^n$.