PMATH 333 Real Analysis, Assignment 1 Due: Fri Feb 2, 11:00 pm

1: In this problem, you will use the rules R1-R9 which define rings and fields, exactly as stated in the lecture notes, along with the rule R0 which states that for every $a \in R$ we have $a \cdot 0 = 0$ and $0 \cdot a = 0$ (this is Part 4 of Properties of Rings). In your solution, your proof should be very detailed, using only one rule at each step in the proof, and explicitly indicating which rule is used at each step. Note, for example, that to prove that $0 + a = a$ you need to use both R3 and R2.

(a) Let R be a ring. Using only the rules R1-R7 which define a ring, together with rule R0 as stated above, prove that for all $a, b, c, d \in R$, if $a + c = 0$ and $ab + d = 0$ then $cb = d$ (this is Part 6 of Properties of Rings).

(b) Let F be a field. Using only the rules R1-R9 which define a field, together with rule R0 as stated above, prove that for all $x, y \in F$, if $x \cdot x = y \cdot y$ then either $x = y$ or $x + y = 0$.

2: In this problem, you will use rules R1-R9 and O1-O5 which define an ordered field, exactly as stated in the lecture notes, along with rule R0, as stated in Problem 1. Your proof should be very detailed, using only one rule at each step in the proof, and explicitly indicating which rule is used at each step.

(a) Let F be an ordered field. Using only the rules R1-R9 and O1-O5 which define an ordered field, together with the rule R0 from Problem 1, prove that for all $x \in F$, if $0 \leq x$ and $x \leq 1$ then $x \cdot x \leq x$.

(b) Let F be an ordered field. Using only the rules R1-R9 and O1-O5 which define an ordered field, together with the rule R0 from Problem 1, prove that for all $x, y \in F$, if $x \leq 0$ and $y \leq 0$ then $0 \leq xy$.

3: In this problem, you can freely use any of the familiar properties which hold in ordered fields (you do not need to explicitly indicate when your proof uses these rules). But you must explicitly indicate each time your proof uses one of the named order properties involving $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$. To be specific, you must clearly indicate each time your proof uses any of the following properties: the Discreteness Property of $\mathbb{Z}$, the Least Upper (or Greatest Lower) Bound Property of $\mathbb{R}$, the Approximation Property of the Supremum or Infimum, the Well-Ordering Property of $\mathbb{Z}$ in $\mathbb{R}$, the Floor and Ceiling Properties of $\mathbb{Z}$ in $\mathbb{R}$, the Archimedian Property of $\mathbb{Z}$ in $\mathbb{R}$, the Density of $\mathbb{Q}$ in $\mathbb{R}$, the Induction Principle in $\mathbb{Z}$, or the Strong Induction Principle in $\mathbb{Z}$.

(a) Let $S = \left\{ \frac{2+(-1)^n}{n} \middle| n \in \mathbb{Z}^+ \right\}$. Find (with proof) $\sup S$ and $\inf S$.

(b) Let A and B be nonempty bounded subsets of $\mathbb{R}$ and let $C = \left\{ x+y \,\middle|\, x \in A, y \in B \right\}$. Prove that C is bounded and that $\sup A + \sup B = \sup C$.

(c) Let S be a nonempty set in $\mathbb{R}$ which is bounded above. Suppose there exists $\delta > 0$ such that for all $x, y \in S$, if $x \neq y$ then $|y - x| \geq \delta$. Prove that S has a maximum element.

4: Let $n \in \mathbb{Z}^+$ and let $0 < y \in \mathbb{R}$. In this problem you will prove that y has a unique positive n^{th} root $x \in \mathbb{R}$. You can freely use any of the rules and properties discussed in Chapter 1 (you do not need to explicitly indicate when the various rules and properties are being used).

(a) Show that for all $a, b \in \mathbb{R}$, if $0 < a < b$ then $b^n - a^n \leq n\, b^{n-1}(b - a)$.

(b) Let $A = \left\{ 0 < t \in \mathbb{R} \,\middle|\, t^n < y \right\}$. Show that A is nonempty and bounded above and let $x = \sup A$.

(c) Using Part (a), or otherwise, show that $x^n \geq y$, where x is as in Part (b).

(d) Using Part (a), or otherwise, show that $x^n \leq y$.

(e) Finally, show that if x_1 and x_2 are positive real numbers with ${x_1}^n = {x_2}^n$ then $x_1 = x_2$.