MATH 247 Calculus 3 (Advanced), Solutions to the Midterm Test, Winter 2024

1: (a) Define $f:\mathbb{R}\to\mathbb{R}^2$ by $f(t)=\left(t^2\,,\,\frac{t}{t^2+1}\right)$. Find (with proof) $g:\mathbb{R}^2\to\mathbb{R}$ such that $\text{Range}(f)=\text{Null}(g)$ then prove that $\text{Range}(f)$ is closed.

Solution: Define $g:\mathbb{R}^2\to\mathbb{R}$ by $g(x,y)=y^2(x+1)^2-x$. We claim that $\text{Range}(f)=\text{Null}(g)$. Suppose $(x,y)\in\text{Range}(f)$. Choose $t\in\mathbb{R}$ so that $(x,y)=f(t)=\left(t^2\,,\,\frac{t}{t^2+1}\right)$, that is $x=t^2$ and $y=\frac{t}{t^2+1}$. Then we have $g(x,y)=y^2(x+1)^2-x=\left(\frac{t}{t^2+1}\right)^2\left(t^2+1\right)^2-t^2=0$, and so $(x,y)\in\text{Null}(g)$.

Suppose that $(x,y)\in\text{Null}(g)$ so that we have $0=g(x,y)=y^2(x^2+1)^2-x$, that is $y^2(x+1)^2=x$. Let $t=y(x+1)$. Then we have $t^2=y^2(x+1)^2=x$ and $\frac{t}{t^2+1}=\frac{y(x+1)}{x^2+1}=y$, so that $(x,y)=f(t)$, and hence $(x,y)\in\text{Range}(f)$. Thus $\text{Range}(f)=\text{Null}(g)$, as claimed.

Finally, note that $\text{Null}(g)$ is closed in $\mathbb{R}^2$ because $g:\mathbb{R}^2\to\mathbb{R}$ is continuous (it is a polynomial) and $\{0\}$ is closed in $\mathbb{R}$, and we have $\text{Null}(g)=g^{-1}\left(\{0\}\right)$.

(b) Define $g:\mathbb{R}^3\to\mathbb{R}^2$ by $g(x,y,z)=\left(x^2+y^2+z\,,\,2x+z\right)$. Find $f:\mathbb{R}\to\mathbb{R}^3$ such that $\text{Range}(f)=g^{-1}(1,1)$ then prove that $g^{-1}(1,1)$ is connected.

Solution: Define $f:\mathbb{R}\to\mathbb{R}^2$ by $f(t)=\left(1+\cos t\,,\,\sin t\,,\,-1-\cos t\right)$. We claim that $\text{Range}(f)=g^{-1}(1,1)$.

Let $(x,y,z)\in g^{-1}(1,1)$, so we have $x^2+y^2+z=1$ and $2x+z=1$. Then $z=1-x^2-y^2=1-2x$ so that $x^2-2x+y^2=0$, that is $(x-1)^2+y^2=1$. Since (x,y) is on the circle of radius 1 centred at $(1,0)$, we can choose $t\in\mathbb{R}$ such that $(x,y)=(1+\cos t,\sin t)$, and then $z=1-2x=1-2(1+\cos t)=-1-2\cos t$. This shows that $g^{-1}(1,1)\subseteq\text{Range}(f)$.

Suppose, conversely, that $(x,y,z)\in\text{Range}(f)$, say $(x,y,z)=f(t)=\left(1+\cos t,\sin t,-1-2\cos t\right)$. Then we have $x^2+y^2+z=(1+\cos t)^2+(\sin t)^2-1-2\cos t=1+2\cos t+\cos^2 t+\sin^2 t-1-2\cos t=1$ and $2x+z=2(1+\cos t)-1-2\cos t=1$ so that $g(x,y)=(1,1)$, This shows that $\text{Range}(f)\subseteq g^{-1}(1,1)$.

Finally, we claim that $\text{Range}(f)$ is path-connected (hence connected). Let $a,b\in\text{Range}(f)$, say $a=f(r)$ and $b=f(s)$. Then the map $\alpha:[0,1]\to\text{Range}(f)$ given by $\alpha(t)=f(a+t(b-a))$ is a continuous path from a to b in $\text{Range}(f)$. Thus $\text{Range}(f)$ is path-connected, as claimed.

2: (a) Let $A = \big\{(x,y)\in\mathbb{R}^2\,\big|\,3x+2y<6\big\}$. Prove, from the definition of an open set, that A is open in $\mathbb{R}^2$.

Solution: Let $(a,b)\in A$, so we have $3a+2b<6$. Let $r=\frac{1}{5}\big(6-(3a+2b)\big)>0$ and note that $3a+2b=6-5r$. When $(x,y)\in B\big((a,b),r\big)$ we have $x-a\le|x-a|=\sqrt{(x-a)^2}\le\sqrt{(x-a)^2+(y-b)^2}<r$, and similarly we have $y-b<r$, and hence $3x+2y<3(a+r)+2(b+r)=3a+2b+5r=6$, so that $(x,y)\in A$.

(b) For $n\ge1$, let $s_n=\sum\limits_{k=1}^{n}\left(\frac{1+i}{3}\right)^k$. Prove, from the definition of a limit, that $\lim\limits_{n\to\infty}s_n=\frac{1+3\,i}{5}$.

Solution: Using the formula for the sum of a geometric series, we have

$$s_n=\sum_{k=1}^{n}\left(\tfrac{1+i}{3}\right)^k=\frac{\left(\frac{1+i}{3}\right)\left(1-\left(\frac{1+i}{3}\right)^n\right)}{1-\left(\frac{1+i}{3}\right)}=\frac{\left(\frac{1+i}{3}\right)\left(1-\left(\frac{1+i}{3}\right)^n\right)}{\frac{2-i}{3}}\cdot\frac{2+i}{2+i}=\frac{\frac{1+3i}{3}\left(1-\left(\frac{1+i}{3}\right)^n\right)}{\frac{5}{3}}=\frac{1+3i}{5}\left(1-\left(\tfrac{1+i}{3}\right)^n\right)$$

so that

$$\left|s_n-\tfrac{1+3i}{5}\right|=\left|\tfrac{1+3i}{5}\left(\tfrac{1+i}{3}\right)^n\right|=\left|\tfrac{1+3i}{5}\right|\cdot\left|\tfrac{1+i}{3}\right|^n=\sqrt{\tfrac{2}{5}}\cdot\left(\tfrac{\sqrt{2}}{3}\right)^n.$$

Given $\epsilon>0$ we choose $m\in\mathbb{Z}^+$ so that $\sqrt{\frac{2}{5}}\cdot\left(\frac{\sqrt{2}}{3}\right)^m<\epsilon$, and then for $n\ge m$ we have $\left|s_n-\frac{1+3i}{5}\right|<\epsilon$.

(c) Let $f(x)=\frac{xy}{x^2+y^2}$. Prove, from the definition of a limit, that $\lim\limits_{(x,y)\to(0,0)}f(x,y)$ does not exist.

Solution: Suppose, for a contradiction, that $\lim\limits_{(x,y)\to(0,0)}f(x,y)$ does exist, and let $b=\lim\limits_{(x,y)\to(0,0)}f(x,y)$. Taking $\epsilon=\frac{1}{2}$, we can choose $\delta>0$ such that for all (x,y), if $0<\sqrt{x^2+y^2}<\delta$ then $\big|f(x,y)-b\big|<\frac{1}{2}$. When $(x,y)=\left(\frac{\delta}{2},\frac{\delta}{2}\right)$ we have $0<\sqrt{x^2+y^2}=\frac{\delta}{\sqrt{2}}<\delta$ and we have $f(x,y)=\frac{1}{2}$, and hence $\left|\frac{1}{2}-b\right|<\frac{1}{2}$, which implies that $0<b<1$. On the other hand, when $(x,y)=\left(-\frac{\delta}{2},\frac{\delta}{2}\right)$, we have $0<\sqrt{x^2+y^2}=\frac{\delta}{\sqrt{2}}<\delta$ and we have $f(x,y)=-\frac{1}{2}$, and hence $\left|-\frac{1}{2}-b\right|<\frac{1}{2}$, which implies that $-1<b<0$. This gives the desired contradiction (since we cannot have $-1<b<0$ and $0<b<1$).

3: (a) Let $A \subseteq \mathbb{R}^n$. Prove that $\overline{A} = A \cup A'$ (this is part of Theorem 2.19).

Solution: We shall show that $A \cup A'$ is the smallest closed set which contains A. We claim that $A \cup A'$ is closed. Let $a \in (A \cup A')^c$. Since $a \notin A'$ we can choose $r > 0$ such that $B(a,r) \cap A = \emptyset$. Suppose, for a contradiction, that $B(a,r) \cap A' \neq \emptyset$, say $b \in B(a,r) \cap A'$. Since $b \in B(a,r)$, which is open, we can choose $s > 0$ such that $B(b,s) \subseteq B(a,r)$, and since $b \in A'$ we have $B(a,s) \cap A \neq \emptyset$. Since $B(a,s) \subseteq B(a,r)$ and $B(a,s) \cap A \neq \emptyset$, it follows that $B(a,r) \cap A \neq \emptyset$, giving the desired contradiction. Thus we have $B(a,r) \cap A' = \emptyset$. Since $B(a,r) \cap A = \emptyset$ and $B(a,r) \cap A' = \emptyset$, we have $B(a,r) \subseteq (A \cup A')^c$. Thus $A \cup A'$ is closed.

It remains to show that for every closed set $K \subseteq \mathbb{R}^n$ with $A \subseteq K$ we have $A \cup A' \subseteq K$. Let $K \subseteq \mathbb{R}^n$ be closed with $A \subseteq K$. Since K is closed we have $K = K'$. Since $A \subseteq K$ we have $A' \subseteq K' = K$. Since $A \subseteq K$ and $A' \subseteq K$ we have $A \cup A' \subseteq K$, as required.

(b) Let $A = \left\{ (u,v,w,x,y,z) \in \mathbb{R}^6 \,\middle|\, \text{rank}\left(\begin{smallmatrix} u & v & w \\ x & y & z \end{smallmatrix}\right) < 2 \right\}$. Determine whether A is closed and whether A is compact.

Solution: Note that we have $\text{rank}\left(\begin{smallmatrix} u & v & w \\ x & y & z \end{smallmatrix}\right) = 2$ if and only if some pair of columns is linearly independent if and only if one of the three 2×2 sub-matrices $\left(\begin{smallmatrix} u & v \\ x & y \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} u & w \\ x & z \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} v & w \\ y & z \end{smallmatrix}\right)$ is invertible if and only if one of the three determinants $uy - vx$, $uz - wx$ and $vz - wy$ is non-zero. Thus we have

$$\text{rank}\left(\begin{smallmatrix} u & v & w \\ x & y & z \end{smallmatrix}\right) < 2 \iff \Big(uy - vx = 0 \text{ and } uz - wx = 0 \text{ and } vz - wy = 0 \Big)$$

and hence $A = f^{-1}\big(\{0\}\big) \cap g^{-1}\big(\{0\}\big) \cap h^{-1}\big(\{0\}\big)$ where $f, g, h : \mathbb{R}^6 \to \mathbb{R}$ are given by

$$f(u,v,w,x,y,z) = uy - vx \,,\ \ g(u,v,w,x,y,z) = uz - wx \,,\ \text{and}\ \ h(u,v,w,x,y,z) = vz - wy \,.$$

Since f, g and h are continuous (they are polynomials) and $\{0\}$ is closed in $\mathbb{R}$, it follows that the sets $f^{-1}\big(\{0\}\big)$, $g^{-1}\big(\{0\}\big)$ and $h^{-1}\big(\{0\}\big)$ are all closed, and hence the set A is closed. On the other hand, A is not bounded because for $e_1 = (1,0,0,0,0,0)$ we have $re_1 \in A$ for all $r \in \mathbb{R}$ and $\|re_1\| = |r|$. Since A is not bounded, it is not compact (by the Heine-Borel Theorem).

4: (a) Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ be continuous. Show that if A is non-empty and compact then f attains its maximum value on A (this is part of Theorem 3.37).

Solution: suppose that $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ with A is compact. Since A is compact and f is continuous, $f(A)$ is compact by Part (2). Since $f(A)$ is compact, it is closed and bounded by the Heine Borel Theorem. Since $f(A)$ is bounded and non-empty (since $A \neq \emptyset$) it has a supremum and an infemum in $\mathbb{R}$. Let $u = \sup f(A)$. By the Approximation Property of the Supremum, for each $n \in \mathbb{Z}^+$ we can choose $x_n \in A$ with $u - \frac{1}{n} < f(x_n) \leq u$, and it follows that $f(x_n) \to u$ and hence u is a limit point of $f(A)$. Since u is a limit point of $f(A)$ and $f(A)$ is closed, we have $u \in f(A)$. Thus we can choose $a \in A$ such that $f(a) = u = \sup f(A) = \max f(A)$, and then f attains its maximum value at $a \in A$.

(b) Let $A, B \subseteq \mathbb{R}^n$. Show that if A is compact, B is closed and $A \cap B = \emptyset$, then there exists $r > 0$ such that the open sets $U = \bigcup_{a \in A} B(a, r)$ and $V = \bigcup_{b \in B} B(b, r)$ are disjoint.

Solution: Suppose that A is compact, B is closed and $A \cap B = \emptyset$. Since $A \cap B = \emptyset$ we have $A \subseteq B^c$. For each $a \in A$, since $a \in B^c$ and B^c is open, we can choose $r_a > 0$ such that $B(a, 3r_a) \subseteq B^c$. Note that the set $S = \{B(a, r_a) \,|\, a \in A\}$ is an open cover of A. Since A is compact, we can choose a finite subcover of S, so we can choose $a_1, a_2, \cdots, a_\ell \in A$ such that $A \subseteq B(a_1, r_{a_1}) \cup \cdots \cup B(a_\ell, r_{a_\ell})$. Let $r = \min\{r_{a_1}, \cdots, r_{a_\ell}\}$.

Let $U = \bigcup_{a \in A} B(a, r)$ and $V = \bigcup_{b \in B} B(b, r)$. We claim that $U \cap V = \emptyset$. Suppose, for a contradiction, that $U \cap V \neq \emptyset$ and choose $x \in U \cup V$. Since $x \in U$ we can choose $a \in A$ such that $x \in B(a, r)$, since $a \in A$ we choose k so that $a \in B(a_k, r_k)$, and since $x \in V$ we can choose $b \in B$ so that $x \in B(b, r)$. But then we have $|b - a_k| \leq |b - x| + |x - a| + |a - a_k| < r + r + r_k \leq 3r_{a_k}$ so that $b \in B(a_k, 3r_{a_k})$. This is not possible since $b \in B$ and $B(a_k, 3r_{a_k}) \subseteq B^c$.