MATH 247 Calculus 3, Solutions to the Exercises for Chapter 6

1: (a) A function $f(x,y)$ is called **harmonic** if it is a solution to **Laplace's equation**, which is the partial differential equation $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$. Determine which of the following two functions are harmonic.

(i) $f(x,y)=\ln\sqrt{x^2+y^2}$

Solution: We have $f(x,y)=\frac{1}{2}\ln(x^2+y^2)$, so $\frac{\partial f}{\partial x}=\frac{x}{x^2+y^2}$ and $\frac{\partial f}{\partial y}=\frac{y}{x^2+y^2}$ and so $\frac{\partial^2 f}{\partial x^2}=\frac{-x^2+y^2}{(x^2+y^2)^2}$ and $\frac{\partial^2 f}{\partial y^2}=\frac{x^2-y^2}{(x^2+y^2)^2}$. Since $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$, $f(x,y)$ is harmonic.

(ii) $f(x,y)=\tan^{-1}\frac{y}{x}$.

Solution: We have $\frac{\partial f}{\partial x}=\frac{-\frac{y}{x}}{1+\frac{y^2}{x^2}}=-\frac{y}{x^2+y^2}$ and $\frac{\partial f}{\partial y}=\frac{\frac{1}{x}}{1+\frac{y^2}{x^2}}=\frac{x}{x^2+y^2}$ and so we have $\frac{\partial^2 f}{\partial x^2}=\frac{2xy}{(x^2+y^2)^2}$ and $\frac{\partial^2 f}{\partial y^2}=\frac{-2xy}{(x^2+y^2)^2}$. Since $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$, $f(x,y)$ is harmonic.

(b) Find the Taylor polynomial of degree 2, centred at $(-2,1)$, for $f(x,y)=(2-x)\,e^{x+2y}$.

Solution: We have

$$\frac{\partial f}{\partial x}=(1-x)e^{x+2y}\ ,\ \frac{\partial f}{\partial y}=2(2-x)e^{x+2y}\ ,\ \frac{\partial^2 f}{\partial x^2}=(-x)e^{x+2y}\ ,\ \frac{\partial^2 f}{\partial x\partial y}=2(1-x)e^{x+2y}\ ,\ \frac{\partial^2 f}{\partial y^2}=4(2-x)e^{x+2y}$$

and so

$$f(-2,1)=4\ ,\ \frac{\partial f}{\partial x}(-2,1)=3\ ,\ \frac{\partial f}{\partial y}(-2,1)=8\ ,\ \frac{\partial^2 f}{\partial x^2}(-2,1)=2\ ,\ \frac{\partial^2 f}{\partial x\partial y}(-2,1)=6\ ,\ \frac{\partial^2 f}{\partial y^2}(-2,1)=16.$$

Thus the second Taylor polynomial is

$$T_2(x,y)=4+3(x+2)+8(y-1)+(x+2)+6(x+2)(y-1)+8(y-1)^2.$$

2: (a) Let $z = f(x,y) = x^2y + 2x^2 + y^2$. Find and classify all the critical points of $f(x,y)$, then find the maximum and minimum values of $z = f(x,y)$ in $D = \{(x,y)|x^2+y^2 \le 8\}$.

Solution: We have $Df(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} 2xy+4x & x^2+2y \end{pmatrix}$. Note that

$$\frac{\partial f}{\partial x} = 0 \iff 2xy+4x = 0 \iff 2x(y+2) = 0 \iff x = 0 \text{ or } y = -2.$$

When $x = 0$ we have $x^2 + 2y = 0 \iff y = 0$, and when $y = -2$ we have $x^2+2y = 0 \iff x = \pm 2$. Thus we have $Df(x,y) = 0$ at the points $(0,0)$, $(2,-2)$ and $(-2,-2)$. The Hessian matrix is

$$Hf(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 2y+4 & 2x \\ 2x & 2 \end{pmatrix}.$$

When $(x,y) = (0,0)$, we have $Hf = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$, which has eigenvalues 4 and 2, so f has a local minimum at $(0,0)$. When $(x,y) = (2,-2)$, we have $Hf = \begin{pmatrix} 0 & 4 \\ 4 & 2 \end{pmatrix}$ and $\det\big(Hf - \lambda I\big) = \det\begin{pmatrix} -\lambda & 4 \\ 4 & \lambda \end{pmatrix} = \lambda^2 - 2\lambda - 16$, so Hf has eigenvalues $\lambda = \frac{2 \pm \sqrt{4+4\cdot 16}}{2} = 1 \pm \sqrt{17}$, and so f has a saddle point at $(-2,2)$. When $(x,y) = (-2,-2)$, we have $Hf = \begin{pmatrix} -0 & -4 \\ -4 & 2 \end{pmatrix}$, which also has eigenvalues $1 \pm \sqrt{17}$, so f also has a saddle point at $(-2,-2)$.

On the boundary, where $x^2 = 8 - y^2$ with $-2\sqrt{2} \le y \le 2\sqrt{2}$, we have

$$f(x,y) = (8-y^2)y + 2(8-y^2) + y^2 = -y^3 - y^2 + 8y + 16$$

so we let $g(y) = -y^3 - y^2 + 8y + 16$. Then $g'(y) = -3y^2 - 2y + 8 = -(3y-4)(y+2)$, so $g'(y) = 0$ when $y = \frac{4}{3}$ and -2. When $y = \frac{4}{3}$ we have $x = \pm\sqrt{8 - \frac{16}{9}} = \pm\frac{2\sqrt{14}}{3}$, when $y = -2$ we have $x = \pm 2$, and when $y = \pm 2\sqrt{2}$ we have $x = 0$. Note that $f(0,0) = 0$, and $f(\pm 2, -2) = g(-2) = 4$, and $f\big(\pm\frac{2\sqrt{14}}{3}, \frac{4}{3}\big) = g\big(\frac{4}{3}\big) = \frac{608}{27}$, and $f\big(\pm 2\sqrt{2}, 0\big) = g(\pm 2\sqrt{2}) = 8$, so the minimum value of f is $f(0,0) = 0$ and the maximum value of f is $f\big(\pm\frac{\sqrt{14}}{3}, \frac{4}{3}\big) = \frac{608}{27}$.

(b) Find the maximum possible area for a quadrilateral with vertices at $(0,0)$, $(1-r,0)$, $(1-r+r\cos\theta, r\sin\theta)$ and $(0, r\sin\theta)$, with $0 \le r \le 1$ and $0 \le \theta \le \frac{\pi}{2}$.

Solution: The area is given by

$$A(r,\theta) = (1-r)(r\sin\theta) + \tfrac{1}{2}(r\sin\theta)(r\cos\theta) = r\sin\theta - r^2\sin\theta + \tfrac{1}{2}r^2\sin\theta\cos\theta$$

and we have

$$DA(r,\theta) = \big(\tfrac{\partial A}{\partial r}, \tfrac{\partial A}{\partial \theta}\big) = \big(\sin\theta - 2r\sin\theta + r\sin\theta\cos\theta, r\cos\theta - r^2\cos\theta + \tfrac{1}{2}r^2\cos^2\theta - \tfrac{1}{2}r^2\sin^2\theta\big).$$

Let us find the critical points with $0 < r < 1$ and $0 < \theta < \frac{\pi}{2}$. We have $DA(r,\theta) = 0$ when

$$\sin\theta(1 - 2r + r\cos\theta) = 0 \quad (1) \quad \text{and} \quad r\cos\theta - r^2\cos\theta + r^2\cos^2\theta - \tfrac{1}{2}r^2 = 0 \quad (2).$$

For $0 < \theta < \frac{\pi}{2}$ we have $\sin\theta \ne 0$, so equation (1) gives $1 - 2r + r\cos\theta = 0$, that is $\cos\theta = \frac{2r-1}{r}$. Put this into equation (2) to get $(2r-1) - r(2r-1) + (2r-1)^2 - \frac{1}{2}r^2 = 0$, that is $\frac{3}{2}r^2 - r = 0$. For $r > 0$ this gives $r = \frac{2}{3}$, and since $\cos\theta = \frac{2r-1}{r} = \frac{1}{2}$, we have $\theta = \frac{\pi}{3}$. Thus the only critical point is at $\big(\frac{2}{3}, \frac{\pi}{2}\big)$, and we have $A\big(\frac{2}{3}, \frac{\pi}{3}\big) = \frac{\sqrt{3}}{9} + \frac{\sqrt{3}}{18} = \frac{\sqrt{3}}{6}$.

On the boundary, when $r = 0$ we have $A = 0$, when $r = 1$ we have $A = \frac{1}{2}\sin\theta\cos\theta = \frac{1}{4}\sin 2\theta \le \frac{1}{4} < \frac{\sqrt{3}}{6}$, when $\theta = 0$ we have $A = 0$, and when $\theta = \frac{\pi}{2}$ we have $A = (1-r)r = \frac{1}{4} - \big(r - \frac{1}{2}\big)^2 \le \frac{1}{4} < \frac{\sqrt{3}}{6}$. Thus the maximum possible area is $A = \frac{\sqrt{3}}{6}$, which we obtain when $(r,\theta) = \big(\frac{2}{3}, \frac{\pi}{3}\big)$.

3: Let $u = f(x,y,z) = x^2 + xy + y^2 + 3yz^2 + 6z^2$. Find and classify all the critical points of $f(x,y,z)$, then find the maximum and minimum values of u with $-1 \le x \le 3$, $-4 \le y \le 0$ and $z = 1$.

Solution: We have

$$Df(x,y) = \left(\tfrac{\partial u}{\partial x}, \tfrac{\partial u}{\partial y}, \tfrac{\partial u}{\partial z}\right) = \left(2x+y,\, x+2y+3z^2,\, 6yz+12z\right)$$

and so $Df = 0$ when

$$2x + y = 0 \quad (1)\ ,\ \ x + 2y + 3z^2 = 0 \quad (2)\ ,\ \ \text{and}\ \ 6z(y+2) = 0 \quad (3).$$

From (3) we have $z = 0$ or $y = -2$. When $z = 0$, equation (2) becomes $x + 2y = 0$, and this, together with equation (1), implies $(x,y) = (0,0)$. When $y = -2$, equation (1) gives $x = 1$, and then equation (2) becomes $1 - 4 + 3z^2 = 0$, so $z = \pm 1$. Thus the critical points are $(0,0,0)$, $(1,-2,1)$ and $(1,-2,-1)$.

The Hessian matrix is

$$Hf = \begin{pmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} & \frac{\partial^2 u}{\partial x \partial z} \\ \frac{\partial^2 u}{\partial y \partial x} & \frac{\partial^2 u}{\partial y^2} & \frac{\partial^2 u}{\partial y \partial z} \\ \frac{\partial^2 u}{\partial z \partial x} & \frac{\partial^2 u}{\partial z \partial y} & \frac{\partial^2 u}{\partial z^2} \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 6z \\ 0 & 6z & 12 \end{pmatrix}.$$

At $(0,0,0)$ we have

$$Hf = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 12 \end{pmatrix}$$

and $\det\left(Hf - \lambda I\right) = ((2-\lambda)^2 - 1)(12-\lambda) = (12-\lambda)(\lambda^2 - 4\lambda + 3) = -(\lambda-12)(\lambda-3)(\lambda-1)$ so Hf has eigenvalues 1, 3 and 12, and so u has a local minimum at $(0,0,0)$. At $(1,-2,\pm 1)$, we have

$$Hf = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & \pm 6 \\ 0 & \pm 6 & 12 \end{pmatrix}$$

and $\det\left(Hf - \lambda I\right) = (2-\lambda)^2(12-\lambda) - 72 - 12 = -\lambda^3 + 16\lambda^2 - 52\lambda - 36 = g(\lambda)$, say. Note that $\lim_{\lambda \to -\infty} g(\lambda) = \infty > 0$, $g(0) = -36 < 0$, $g(10) = 414 > 0$ and $\lim_{\lambda \to \infty} g(\lambda) = -\infty < 0$, so by the Intermediate Value Theorem, $g(\lambda) = 0$ has a solution in each of the intervals $(-\infty, 0)$, $(0,10)$, and $(10,\infty)$. Thus Hf has one negative eigenvalue and two positive eigenvalues, so u has a saddle point at $(1,-2,\pm 1)$.

When $z = 1$ we have $u = x^2 + xy + y^2 + 3y + 6 = h(x,y)$, say. We have $Dh(x,y) = \left(2x+y,\, x+2y+3\right)$, so $Dh(x,y) = 0$ when $2x + y = 0$ and $x + 2y + 3 = 0$, that is when $(x,y) = (1,-2)$. The Hessian matrix for h is

$$Hh = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

and $\det\left(Hh - \lambda I\right) = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda-1)(\lambda-3)$, so Hh has eigenvalues 1 and 3, and so u has a local minimum at $(x,y) = (1,-2)$ with $u(1,-2,1) = h(1,-2) = 1 - 2 + 4 - 6 + 6 = 3$.

On the edges of the boundary, when $x = -1$ we have

$$u = 1 - y + y^2 + 3y + 6 = y^2 + 2y + 7 = (y+1)^2 + 6$$

which is a parabola with a minimum of 6 when $y = -1$, and when $x = 3$ we have

$$u = 9 + 3y + y^2 + 3y + 6 = y^2 + 6y + 15 = (y+3)^2 + 6$$

which is a parabola with a minimum of 6 when $y = -3$, and when $y = -4$ we have

$$u = x^2 - 4x + 16 - 12 + 6 = x^2 - 4x + 10 = (x-2)^2 + 6$$

which is a parabola with a minimum of 6 at $x = 2$, and when $y = 0$, we have $u = x^2 + 6$ which is a parabola with a minimum of 6 at $x = 0$.

At the corner points of the boundary, we have $u(-1,-4,1) = 15$, $u(-1,0,1) = 7$, $u(3,-4,1) = 7$ and $u(3,0,1) = 15$. Thus the maximum value of u is 15, which occurs at the corners $(-1,0,1)$ and at $(3,0,1)$, and the minimum value is 3, which occurs at the interior point $(1,-2,1)$.