MATH 247 Calculus 3, Solutions to the Exercises for Chapter 1

1: Let $A = \text{Range}(f)$ where $f : \mathbf{R} \to \mathbf{R}^2$ is given by $f(t) = \big(\cos t, \sin 2t\big)$ and let $B = \text{Null}(g)$ where $g : \mathbf{R}^2 \to \mathbf{R}$ is given by $g(x, y) = y^2 + 4x^2(x^2 - 1)$. Show (algebraically) that $A = B$, and then sketch the set $A \subseteq \mathbf{R}^2$ (it is a curve in $\mathbf{R}^2$).

Solution: Note that $A = \text{Range}(f) = \big\{(\cos t, \sin 2t)\big| t \in \mathbf{R}\big\}$ and $B = \text{Null}(g) = \big\{(x, y)\big| y^2 + 4x^2(x^2 - 1) = 0\big\}$. Let $(x, y) \in A$. Choose $t \in \mathbf{R}$ such that $x = \cos t$ and $y = \sin 2t$. Then $x^2 = \cos^2 t$ and

$$y^2 = 4\sin^2 t\cos^2 t = 4\cos^2 t(1 - \cos^2 t) = 4x^2(1 - x^2)$$

so we have $y^2 + 4x^2(x^2 - 1) = 0$ and so $(x, y) \in B$. Thus $A \subseteq B$.

Conversely, suppose that $(x, y) \in B$ so we have $y^2 = 4x^2(1 - x^2)$. Then $y = \pm 2x\sqrt{1 - x^2}$ with $-1 \le x \le 1$. If $y = 2x\sqrt{1 - x^2}$ then we can let $t = \cos^{-1} x \in [0, \pi]$, and then $\cos t = x$ and, since $\sin t \ge 0$,

$$\sin 2t = 2\sin t\cos t = 2\cos t\sqrt{\sin^2 t} = 2\cos t\sqrt{1 - \cos^2 t} = 2x\sqrt{1 - x^2} = y\,.$$

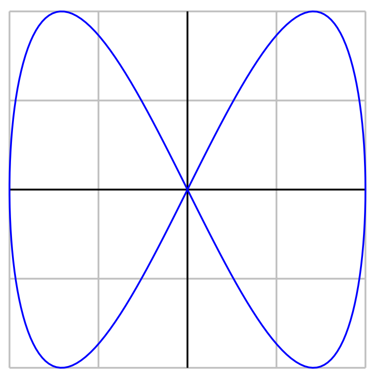
If $y = -2x\sqrt{1 - x^2}$ then we can let $t = -\cos^{-1} x \in [-\pi, 0]$, and then $\cos t = x$ and, since $\sin t \le 0$,

$$\sin 2t = 2\sin t\cos t = -2\cos t\sqrt{\sin^2 t} = -2\cos t\sqrt{1 - \cos^2 t} = -2x\sqrt{1 - x^2} = y\,.$$

In either case, we can choose $t \in \mathbf{R}$ such that $(x, y) = (\cos t, \sin 2t)$ and so $(x, y) \in A$. Thus $B \subseteq A$.

To sketch the set A, we make a table of values and plot points.

t	x	y
0	1	0
$\pi/6$	$\sqrt{3}/2$	$\sqrt{3}/2$
$\pi/4$	$\sqrt{2}/2$	1
$\pi/3$	$1/2$	$\sqrt{3}/2$
$\pi/2$	0	0
$2\pi/3$	$-1/2$	$-\sqrt{3}/2$
$3\pi/4$	$-\sqrt{2}/2$	-1
$5\pi/6$	$-\sqrt{3}/2$	$-\sqrt{3}/2$
π	-1	0
etc		



2: A light, represented by the point $(0, 0, 5)$, lies above the ground, which is represented by the xy-plane. The position of a fly at time $t \ge 0$ is given by $(x, y, z) = (t, t^2, t^3)$. Find the position of the shadow of the fly at time t (you are finding a parametric equation for the curve in the xy-plane traced by the shadow of the fly).

Solution: When the fly is at the point (x, y, z), with $z < 5$, the line from the light at $(0, 0, 5)$ to the fly at (x, y, z) has parametric equation

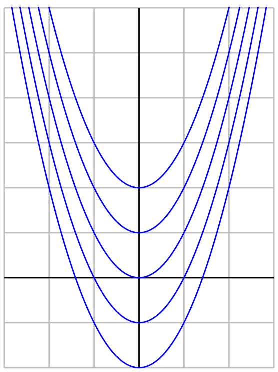
$$(u, v, w) = (0, 0, 5) + s\,\big((x, y, z) - (0, 0, 5)\big) = \big(sx\,,\ sy\,,\ 5 + s(z - 5)\big)\,.$$

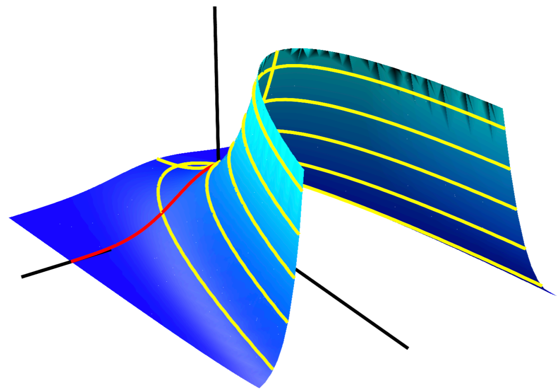
To get $w = 0$, we need $5 + s(z - 5) = 0$, and so $s = \dfrac{5}{5 - z}$, and then $u = sx = \dfrac{5x}{5 - z}$ and $v = st = \dfrac{5y}{5 - z}$. This shows that when the fly is at the point $(x, y, z) = (t, t^2, t^3)$, the shadow is at the point

$$(u(t), v(t)) = \left(\frac{5x}{5 - z}, \frac{5y}{5 - z}\right) = \left(\frac{5t}{5 - t^3}, \frac{5t^2}{5 - t^3}\right).$$

3: Let $f(x,y)=2^{y-x^2}$. Sketch the level sets $z=\frac{1}{4},\frac{1}{2},1,2,4$ and the level sets $x=0$ and $y=0$, and then sketch the surface $z=f(x,y)$ (the graph of f).

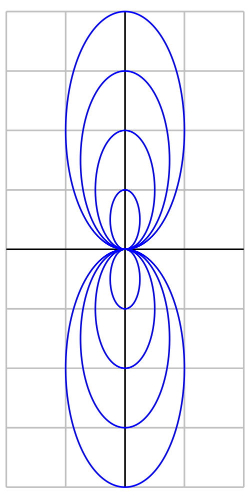
Solution: The level curve $z=\frac{1}{4}$ is the curve $2^{y-x^2}=\frac{1}{4}=2^{-2}$, or equivalently $y-x^2=-2$, that is $y=x^2-2$. Similarly, the level curves $z=\frac{1}{2},1,2,4$ are the parabolas $y=x^2-1$, $y=x^2$, $y=x^2+1$ and $y=x^2+2$, respectively. Also, to find the curve of intersection of the surface with the yz-plane, put in $x=0$ to get the level curve $z=2^y$ (this is shown below in yellow), and to find the curve of intersection with the xz-plane, put in $y=0$ to get the curve $z=2^{-x^2}$ (this is shown below in red).

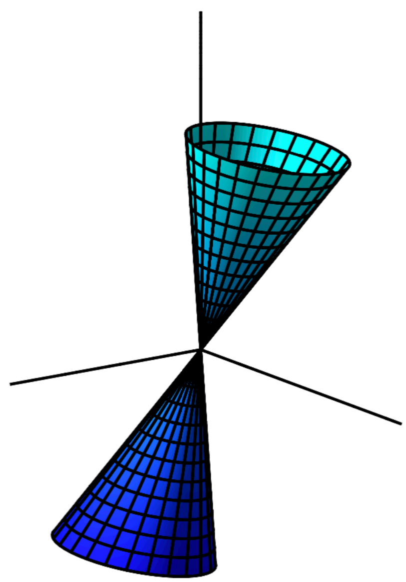




4: Let $f(x,y,z)=4x^2+y^2-yz$. Sketch the level sets $z=0,\pm1,\pm2,\pm3,\pm4$ and the level sets $x=0$ and $y=0$, and then sketch the surface $f(x,y,z)=0$ (the null set of f).

Solution: When $z=0$, the level set has equation $4x^2+y^2=0$; it consists of the single point $(0,0)$. When $z=c\neq 0$, the level set has equation $4x^2+y^2-cy=0$ which we can write as $4x^2+(y-\frac{1}{2}c)^2=\frac{1}{4}c^2$, or equivalently as $\dfrac{x^2}{(c/4)^2}+\dfrac{(y-c/2)^2}{(c/2)^2}=1$, so the level sets are ellipses. The level set $x=0$ has equation $y^2-yz=0$, that is $y(y-z)=0$. It consists of the two lines $y=0$ and $z=y$ in the yz-plane. The level set $y=0$ has equation $4x^2=0$; it is the line $x=0$ in the xz-plane (which is the same as the line $y=0$ in the yz-plane).





5: Let $f(x,y)=x^2+2y^2$ and $g(x,y)=4x-y^2$. Find a parametric equation for the curve of intersection of the two surfaces $z=f(x,y)$ and $z=g(x,y)$.

Solution: Set $f(x,y)=g(x,y)$ to get $x^2+2y^2=4x-y^2$, which we can write as $(x-2)^2+3y^2=4$. This is an ellipse, which we can parametrize as $(x,y)=\left(2+2\cos t,\frac{2}{\sqrt{3}}\sin t\right)$. We also need to have $z=4x-y^2=8+8\cos t-\frac{4}{3}\sin^2 t$, so a parametric equation for the curve of intersection is

$$(x,y,z)=\alpha(t)=\left(2+2\cos t,\frac{2}{\sqrt{3}}\sin t,8+8\cos t-\frac{4}{3}\sin^2 t\right).$$

To be rigorous, let us verify that $\text{Range}(\alpha)=\text{Graph}(f)\cap\text{Graph}(g)$. Let $(x,y,z)\in\text{Range}(\alpha)$. Choose $t\in\mathbf{R}$ such that $(x,y,z)=\alpha(t)$, so we have $x=2+2\cos t$, $y=\frac{2}{\sqrt{3}}\sin t$ and $z=8+8\cos t-\frac{4}{3}\sin^2 t$. Then we have

$$f(x,y)=x^2+2y^2=(2+2\cos t)^2+2\left(\frac{2}{\sqrt{3}}\sin t\right)^2=4+8\cos t+4\cos^2 t+\frac{8}{3}\sin^2 t=8+8\cos t-\frac{4}{3}\sin^2 t=z$$

so that $(x,y,z)\in\text{Graph}(f)$, and we have

$$g(x,y)=4x-y^2=4(2+2\cos t)-\left(\frac{2}{\sqrt{3}}\sin t\right)^2=8+8\cos t-\frac{4}{3}\sin^2 t=z$$

so that $(x,y,z)\in\text{Graph}(g)$. Thus $\text{Range}(\alpha)\subseteq\text{Graph}(f)\cap\text{Graph}(g)$.

Let $(x,y,z)\in\text{Graph}(f)\cap\text{Graph}(g)$. Since $(x,y,z)\in\text{Graph}(f)$ we have $z=f(x,y)=x^2+2y^2$, and since $(x,y,z)\in\text{Graph}(g)$ we have $z=g(x,y)=4x-y^2$. It follows that $x^2+2y^2=4x-y^2$, that is $(x-2)^2+3y^2=4$. Since $(x-2)^2=4-3y^2\leq 4$ we have $\left|\frac{x-2}{2}\right|\leq 1$. Since $3y^2=4-(x-2)^2\leq 4$, we have $\left|\frac{\sqrt{3}}{2}y\right|\leq 1$. Let $t\in[0,2\pi)$ be the (unique) angle with $\sin t=\frac{\sqrt{3}}{2}y$ and $\cos t=\frac{x-2}{2}$. Then we have $x=2+2\cos t$, $y=\frac{2}{\sqrt{3}}\sin t$ and $z=g(x,y)=4x-y^2=8+8\cos t-\frac{4}{3}\sin^t$ and so $(x,y,z)=\alpha(t)\in\text{Range}(\alpha)$. Thus $\text{Graph}(f)\cap\text{Graph}(g)\subseteq\text{Range}(\alpha)$.