Chapter 9. Fubini's Theorem and Change of Variables

9.1 Definition: For $\ell \in \{1,2,\cdots,n\}$, the ℓ^{th} **projection map** $p_\ell : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is given by

$$p_\ell(x_1,x_2,\cdots,x_{\ell-1},y,x_\ell,x_{\ell+1},\cdots,x_{n-1}) = (x_1,x_2,\cdots,x_{n-1}).$$

9.2 Theorem: *(Fubini's Theorem for a Rectangle in $\mathbb{R}^n$). Fix $\ell \in \{1,2,\cdots,n\}$. Let $R = [a_1,b_1]\times[a_2,b_2]\times\cdots\times[a_n,b_n] \subseteq \mathbb{R}^n$ and let $S = p_\ell(R) \subseteq \mathbb{R}^{n-1}$. Let $f : R \to \mathbb{R}$ be integrable on R. For each $x \in S$, define $g_x : [a_\ell,b_\ell] \to \mathbb{R}$ by*

$$g_x(y) = f(x_1,\cdots,x_{\ell-1},y,x_\ell,\cdots,x_{n-1})$$

so that $p_\ell(g_x(y)) = x$. Suppose that g_x is integrable on $[a_\ell,b_\ell]$ for every $x \in S$. Define $G : S \to \mathbb{R}$ by

$$G(x) = \int_{y=a_\ell}^{b_\ell} g_x(y)\,dy.$$

Then G is integrable on $[a_\ell,b_\ell]$ and we have

$$\int_R f = \int_S G = \int_S \left(\int_{y=a_\ell}^{b_\ell} g_x(y)\,dy\right) dV = \int_S \left(\int_{y=a_\ell}^{b_\ell} g_x(y)\,dy\right) dx_1 dx_2\cdots dx_{n-1}.$$

Proof: For convenience of notation, we give the proof in the case that $\ell = n$, so we have $S = [a_1,b_1]\times\cdots\times[a_{n-1},b_{n-1}]$, $g_x(y) = f(x,y)$ and $G(x) = \int_{y=a_n}^{b_n} f(x,y)\,dy$, with $x \in \mathbb{R}^{n-1}$, $y \in \mathbb{R}$. Let $\epsilon > 0$. Choose a partition Z of R with $U(f) \le U(f,Z) < U(f)+\epsilon$. The first $n-1$ components $Z_1, Z_2, \cdots, Z_{n-1}$ of Z determine a partition X of S into sub-rectangles S_k with $k \in K = K(X)$, and the last component of Z gives a partition $Y = Z_n = \{y_0,y_1,\cdots,y_m\}$ of $[a_n,b_n]$, and then Z partitions R into the sub-rectangles $R_{k,j} = S_k \times [y_{j-1},y_j]$ with $|R_{k,j}| = |S_k|(y_j - y_{j-1})$. Let $M_{k,j} = \sup\big\{f(x,y)\,\big|\,(x,y)\in R_{k,j}\big\}$ so that $U(f,Z) = \sum\limits_{k\in K}\sum\limits_{j=1}^{m} M_{k,j}|S_k|(y_j - y_{j-1})$.
Note that

$$G(x) = \int_{y=c}^{d} f(x,y)\,dy = \sum_{j=1}^{m} G_j(x) \quad\text{where}\quad G_j(x) = \int_{y=y_{j-1}}^{y_j} f(x,y)\,dy$$

and note that when $(x,y)\in R_{k,j}$ we have $f(x,y) \le M_{k,j}$ so $G_j(x) \le M_{k,j}(y_{j-1} - y_j)$. Also note that for any bounded maps $p,q : S \to \mathbb{R}$ we have $U\big((p+q),X\big) \le U(p,X) + U(q,X)$ because $\sup\big\{p(x)+q(x)\,\big|\,x\in S_k\big\} \le \sup\big\{p(x)\,\big|\,x\in S_k\big\} + \sup\big\{q(x)\,\big|\,x\in S_k\big\}$. Thus we have

$$\begin{aligned} U(G,X) = U\big(\sum_{j=1}^{m} G_j\,,\,X\big) &\le \sum_{j=1}^{m} U(G_j,X) = \sum_{j=1}^{m}\sum_{k\in K} \sup\big\{G_j(x)\,\big|\,x\in S_k\big\}\,|S_k| \\ &\le \sum_{j=1}^{m}\sum_{k\in K} M_{k,j}(y_j - y_{j-1})\,|S_k| = U(f,Z) < U(f)+\epsilon. \end{aligned}$$

Since $U(G) \le U(G,X) < U(f)+\epsilon$ for all $\epsilon > 0$, it follows that $U(G) \le U(f)$. A similar argument shows that $L(G) \ge L(f)$, so we have

$$L(f) \le L(G) \le U(G) \le U(f).$$

Since f is integrable so that $L(f) = U(f)$, it follows that $L(f) = L(G) = U(G) = U(f)$ so that G is integrable on S and $\int_S G = \int_R f$, that is

$$\int_R f = \int_S G = \int_S \left(\int_{y=a_n}^{b_n} f(x,y)\,dy\right) dx_1 dx_2\cdots dx_{n-1}.$$

9.3 Theorem: *(Iterated Integration) Fix $\ell \in \{1, 2, \cdots, n\}$. Let $B \subseteq \mathbb{R}^{n-1}$ be a closed Jordan region. Let $g, h : B \to \mathbb{R}$ be continuous with $g(x) \leq h(x)$ for all $x \in B$. Let*

$$A = \left\{ (x_1, x_2, \cdots, x_{\ell-1}, y, x_\ell, \cdots, x_{n-1}) \in \mathbb{R}^n \,\middle|\, x \in B\,,\ g(x) \leq y \leq h(x) \right\}.$$

Then

(1) A is a Jordan region in $\mathbb{R}^n$, and
(2) when $f : A \to \mathbb{R}$ is continuous, we have

$$\int_A f = \int_B \left(\int_{y=a_\ell}^{b_\ell} f(x_1, \cdots, x_{\ell-1}, y, x_\ell, \cdots, x_{n-1})\, dy \right) dx_1 dx_2 \cdots dx_{n-1}$$

Proof: For notational convenience, we give a proof in the case that $\ell = n$, so we have

$$A = \left\{ (x, y) \,\middle|\, x \in B\,,\ g(x) \leq y \leq h(x) \right\}.$$

Verify, as an exercise that $\partial A = C \cup G \cup H$ where

$$\begin{aligned} C &= \left\{ (x, y) \,\middle|\, x \in \partial B\,,\ g(x) \leq y \leq h(x) \right\}, \\ G &= \left\{ (x, y) \,\middle|\, x \in B\,,\ y = g(x) \right\}, \text{ and} \\ H &= \left\{ (x, y) \,\middle|\, x \in B\,,\ y = h(x) \right\}. \end{aligned}$$

Choose a rectangle S in $\mathbb{R}^{n-1}$ which contains B. Note that B is compact and g and h are continuous, hence bounded, so we can choose an interval $[a, b]$ which contains the range of both g and h, so that the rectangle $R = S \times [a, b]$ contains A.

We claim that $U(C) = 0$. Let $\epsilon > 0$. Since B is Jordan measurable we can choose a partition X for S, into sub rectangles S_k with $k \in K$, such that $U(\partial B, X) \leq \frac{\epsilon}{b-a}$. Let Z be the partition of R into sub-rectangles $R_k = S_k \times [a, b]$. Note that for each $k \in K$, we have $R_k \cap C \neq \emptyset \iff S_k \cap \partial B \neq \emptyset$, and hence

$$U(C) \leq U(C, Z) = \sum_{R_k \cap C \neq \emptyset} |R_k| = \sum_{S_k \cap \partial B \neq \emptyset} |S_k|(b - a) = U(\partial B, X)\,(b - a) \leq \epsilon.$$

Since $U(C) \leq \epsilon$ for all $\epsilon > 0$, it follows that $U(C) = 0$, as claimed.

We claim that $U(G) = U(H) = 0$. Let $\epsilon > 0$. Choose $m \in \mathbb{Z}^+$ so that $\frac{b-a}{m} \leq \frac{\epsilon}{2(U(B)+1)}$ and let $Y = \{y_0, y_1, \cdots, y_m\}$ be the partition of $[a, b]$ into m equal-sized subintervals, each of size $\frac{b-a}{m}$. Since B is compact and g is continuous, hence uniformly continuous, we can choose $\delta > 0$ so that when $x_1, x_2 \in B$ with $|x_1 - x_2| < \delta$, we have $|g(x_1) - g(x_2)| < \frac{b-a}{2m}$. Choose a partition X of S into sub-rectangles S_k with $k \in K$, so that firstly, we have $U(B, X) \leq U(B) + 1$, and secondly, for each k we have $|x_1 - x_2| < \delta$ for all $x_1, x_2 \in S_k$. Let Z be the partition of R determined by X and Y, that is the partition into the sub-rectangles $R_{k,j} = S_k \times [y_{j-1}, y_j]$. Note that when $R_{k,j} \cap G \neq \emptyset$ we have $S_k \cap B \neq \emptyset$, and note that for each k there are at most 2 values of j for which $R_{k,j} \cap G \neq \emptyset$ because, if we had $(x_i, g(x_i)) \in G \cap R_{k,j_i}$ with $j_1 < j_2 < j_3$ then we would have $x_1, x_3 \in B$ with $g(x_3) - g(x_1) \geq \frac{b-a}{m}$. Thus

$$U(G) \leq U(G, Z) = \sum_{R_{k,j} \cap G \neq \emptyset} |S_k|\, \tfrac{b-a}{m} \leq 2 \cdot \sum_{S_k \cap B \neq \emptyset} |S_k|\, \tfrac{b-a}{m} = 2\, U(B, X)\, \tfrac{b-a}{m} \leq \epsilon.$$

Since $U(G) \leq \epsilon$ for all $\epsilon > 0$, we have $U(B) = 0$. The same argument shows that $U(H) = 0$.

Finally, we note that since $\partial A = C \cup G \cup H$, we have $U(\partial A) \leq \partial C \cup \partial G \cup \partial H = 0$ (by Theorem 8.9), and hence A is Jordan measurable. This completes the proof of Part 1.

To prove Part 2, note that by Definition 8.17 (the definition of the integral), when we extend the domain of a function from a Jordan region to a containing rectangle, by defining the function to be zero outside the Jordan region, the original function is integrable if and only if the extended function is integrable, and they have the same integral. Extend the map $f : A \to \mathbb{R}$ by zero to obtain the map $f : R \to \mathbb{R}$ with $f(x, y) = 0$ when $(x, y) \notin A$. By the definition of the integral, this extended map f is integrable on R with $\int_R f = \int_A f$. By Fubini's Theorem, we have $\int_A f = \int_R f = \int_S G$ where $G(x) = \int_{y=a}^b f(x, y)\, dy$, which is integrable on S. When $x \in B$ we have $f(x, y) = 0$ unless $g(x) \le y \le h(x)$, and so $G(x) = \int_{y=a}^b f(x, y)\, dy = \int_{y=g(x)}^{h(x)} f(x, y)\, dy$. When $x \notin B$ we have $f(x, y) = 0$ for all y so that $G(x) = 0$. By the definition of the integral again, since $G(x) = 0$ whenever $x \notin B$ we have $\int_S G = \int_B G$, and so

$$\int_A f = \int_R f = \int_S G = \int_B G = \int_B \left(\int_{y=g(x)}^{h(x)} f(x, y)\, dy \right) dx_1 dx_2 \cdots dx_{n-1}.$$

9.4 Theorem: *(Local Change of Variables). Let $U \subseteq \mathbb{R}^n$ be open and let $g : U \to \mathbb{R}^n$ be $\mathcal{C}^1$ with $\det Dg \ne 0$ on U. Then for every $a \in U$ there exists an open set W with $a \in W \subseteq U$ such that $g(W)$ is open and $g : W \to g(W)$ is bijective and its inverse is $\mathcal{C}^1$, and such that for every Jordan region A with $\overline{A} \subseteq W$ and for every continuous function $f : g(A) \to \mathbb{R}$, we have*

$$\int_{g(A)} f = \int_A (f \circ g) \big| \det Dg \big|.$$

Proof: We begin by noting that given $a \in U$, using the Inverse Function Theorem we can choose an open set W with $a \in W \subseteq U$ such that $g(W)$ is open and $g : W \to g(W)$ is bijective and its inverse is $\mathcal{C}^1$. Later in the proof we shrink W to make the theorem hold.

We claim that if $|R| = \int_{g^{-1}(R)} \big| \det Dg \big|$ for every rectangle R in $g(W)$, then we have $\int_{g(A)} f = \int_A (f \circ g) \big| \det Dg \big|$ for every Jordan measurable set A with $\overline{A} \subseteq U$ and every continuous function $f : g(A) \to \mathbb{R}$. Suppose that $|R| = \int_{g^{-1}(A)} \big| \det Dg \big|$ for every rectangle R in $g(W)$, let A be a Jordan region with $\overline{A} \subseteq U$ and let $f : g(A) \to \mathbb{R}$ be continuous. Note that the functions $f^+ = \frac{|f|+f}{2}$ and $f^- = \frac{|f|-f}{2}$ are both continuous and non-negative with $f = f^+ - f^-$, so it suffices to consider the case that f is non-negative.

Let $\epsilon > 0$. Choose a rectangle R in $\mathbb{R}^n$ with $g(A) \subseteq R$ and choose a partition X of R into sub-rectangles R_k, $k \in K$ such that $U(f, X) \le \overline{U(f) + \epsilon}$ and such that for all k, if $R_k \cap \overline{g(A)} \ne \emptyset$ then $R_k \subseteq g(W)$ (we can do this since $\overline{g(A)}$ is compact and $g(W)$ is open). Recall that to obtain $U(f, X)$, we first extend f by zero to all of R, and then we let $M_k = \sup \big\{ f(y) \big| y \in R_k \big\}$. Note that when $R_k \cap g(A) = \emptyset$ we have $M_k = 0$, and so we have $U(f, X) = \sum_{R_k \cap \overline{g(A)} \ne \emptyset} M_k |R_k| = \sum_{R_k \cap g(A) \ne \emptyset} M_k |R_k|$ with

$$M_k = \sup\{f(y) | y \in R_k\} = \sup\{f(g(x)) | x \in g^{-1}(R_k)\}.$$

Since the set $\{R_k | R_k \cap g(A) \neq \emptyset\}$ is a set of Jordan regions with disjoint interiors which covers $g(A)$, it follows that the set $\{g^{-1}(R_k) | R_k \cap g(A) \neq \emptyset\}$ is a set of Jordan regions with disjoint interiors which covers A. Let $B = \bigcup_{R_k \cap g(A) \neq \emptyset} g^{-1}(R_k)$. We have

$$\begin{aligned}
\int_{g(A)} f + \epsilon &\geq U(f, X) = \sum_{R_k \cap g(A) \neq \emptyset} M_k \, |R_k| = \sum_{R_k \cap g(A) \neq \emptyset} M_k \int_{g^{-1}(R_k)} \big| \det Dg \big| \\
&\geq \sum_{R_k \cap g(A) \neq \emptyset} \int_{g^{-1}(R_k)} (f \circ g) \big| \det Dg \big| = \int_B (f \circ g) \big| \det Dg \big| \\
&\geq \int_A (f \circ g) \big| \det Dg \big|.
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, it follows that $\int_{g(A)} f \geq \int_A (f \circ g) \big| \det Dg \big|$. A similar argument using $L(f, X)$ shows that $\int_{g(A)} f \leq \int_A (f \circ g) \big| \det Dg \big|$. This proves the claim.

We shall now use the claim to prove the theorem by induction on n. When $n = 1$, the theorem holds by the single variable Change of Variables Theorem. Let $n \geq 2$ and suppose, inductively, that the theorem holds in $\mathbb{R}^{n-1}$. Let $a \in U$. Since $\det Dg(a) \neq 0$, by expanding the determinant along the last row, we see that one of the matrices obtained from $Dg(a)$ by removing the n^{th} row and j^{th} column must have non-zero determinant. For notational convenience, suppose that the upper left $(n-1) \times (n-1)$ submatrix of $Dg(a)$ is invertible. Write elements in W as (x, y) with $x \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}$, re-write the given point $a \in W$ as $(a, b) \in W$, and write $g : W \to g(W)$ as $g(x, y) = \big(h(x, y), g_n(x, y)\big)$ with

$$h(x, y) = \big(g_1(x, y), g_2(x, y), \cdots, g_{n-1}(x, y)\big).$$

Define $p : W \to \mathbb{R}^n$ by

$$p(x, y) = \big(h(x, y), y\big)$$

and note that D_p is the matrix obtained from Dg by replacing the last row by $(0, \cdots, 0, 1)$. In particular $\det Dp(a, b)$ is the determinant of the upper left $(n-1) \times (n-1)$ submatrix of $\det Dg(a, b)$, which we are assuming is non-zero. By the Inverse Function Theorem, we can shrink the open set W, if necessary, so that W and $p(W)$ are open with $(a, b) \in W$, and $p : W \to p(W)$ is invertible with p and p^{-1} both $\mathcal{C}^1$. Define $q : p(W) \to \mathbb{R}^n$ by

$$q(u, v) = \big(u, g_n(p^{-1}(u, v))\big)$$

and note that $q(p(x, y)) = g(x, y)$ for all $(x, y) \in W$ so that g is the composite $g = q \circ p$, and $Dp(x, y) = Dq(p(x, y)) Dp(x, y)$ for all $(x, y) \in W$. The sets W, $p(W)$ and $q(p(W)) = g(W)$ are all open, the maps $g : W \to g(W)$, $p : W \to p(W)$ and $q : p(W) \to q(p(W)) = g(W)$ are all bijective, and these maps and their inverses are all $\mathcal{C}^1$.

Let $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$ be a rectangle in $p(W)$. let $S = [a_1, b_1] \times \cdots \times [a_{n-1}, b_{n-1}]$ so that $R = S \times [a_n, b_n]$. For each $y \in [a_n, b_n]$, define $h : S \to \mathbb{R}^{n-1}$ by $h_y(x) = h(x, y)$. By the induction hypothesis, we have $|S| = \int_{{h_y}^{-1}(S)} \big| \det Dh_y \big|$, and so

$$\begin{aligned}
|R| = |S|(b_n - a_n) &= \int_{y=a_n}^{b_n} |S| \, dy = \int_{y=a_n}^{b_n} \int_{{h_y}^{-1}(S)} \big| \det Dh_y \big| \\
&= \int_{y=a_n}^{b_n} \int_{{h_y}^{-1}(S)} \big| \det Dp \big| = \int_{p^{-1}(R)} \big| \det Dp \big|.
\end{aligned}$$

By the claim proven above, it follows that for every Jordan measurable set A with $\overline{A} \subseteq W$ and for every continuous map $f : p(A) \to \mathbb{R}$ we have

$$\int_{p(A)} f = \int_A (f \circ p)\big|\det Dp\big|\,. \qquad (1)$$

We can give a similar argument for the function q. Let $R = S \times I$ with $I = [a_n, b_n]$ be a rectangle in $q(p(W)) = g(W)$. For each $u \in S$ let $k_u : I \to \mathbb{R}$ be given by $k_u(v) = k(u,v) = g_n(p^{-1}(u,v))$. By the single variable Change of Variables Theorem, we have $|I| = \int_{k_u{}^{-1}(I)} \big|\det Dk_u\big|$ and so

$$|R| = |S|\,|I| = \int_S |I| = \int_S \int_{k_u{}^{-1}(I)} \big|\det Dk_u\big| = \int_{k_u{}^{-1}(R)} \big|\det Dk_u\big| = \int_{k_u{}^{-1}(R)} \big|\det Dq\big|.$$

By the claim, it follows that for every Jordan measurable set B with $\overline{B} \subseteq p(W)$ and every continuous map $f : q(B) \to \mathbb{R}$ we have

$$\int_{q(B)} f = \int_B (f \circ q)\big|\det Dq\big|. \qquad (2)$$

Combining (1) and (2), we see that for every Jordan measurable set A with $\overline{A} \subseteq W$ and for every continuous map $f : A \to \mathbb{R}$, letting $b = p(A)$ so that $\overline{B} \subseteq p(W)$, we have

$$\begin{aligned}\int_{g(A)=q(B)} f = \int_{B=p(A)} (f \circ q)\big|\det Dq\big| &= \int_A \Big((f \circ q)\big|\det Dq\big| \circ p\Big)\big|\det Dp\big| \\ &= \int_A ((f \circ q) \circ p)\big|(\det Dq) \circ p\big|\,\big|\det Dp\big| = \int_A (f \circ g)\big|\det Dp\big|.\end{aligned}$$

9.5 Theorem: *(Change of Variables) Let $U \subseteq \mathbb{R}^n$ be open, let $g : U \to \mathbb{R}^n$ be $\mathcal{C}^1$ with $\det Dg \neq 0$ on U, let A be a Jordan region with $\overline{A} \subseteq U$, and let $f : g(A) \to \mathbb{R}$ be continuous. Then*

$$\int_{g(A)} f = \int_A (f \circ g)\big|\det Dg\big|.$$

Proof: I may include a proof later.