Chapter 8. Jordan Content and Integration

8.1 Definition: A (closed, n-dimensional) **rectangle** in $\mathbb{R}^n$ is a set of the form

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] = \Big\{ x \in \mathbb{R}^n \,\Big|\, a_j \leq x_j \leq b_j \text{ for each index } j \Big\}$$

where each $a_j, b_j \in \mathbb{R}$ with $a_j < b_j$. The **size** of the above rectangle R is

$$|R| = \prod_{j=1}^{n} (b_j - a_j).$$

A **partition** X of the above rectangle R consists of a partition $X_j = \{x_{j,0}, x_{j,1}, \cdots, x_{j,\ell_j}\}$ with

$$a_j = x_{j,0} < x_{j,1} < \cdots < x_{j,\ell_k} = b_j$$

for each index j. The above partition X divides the rectangle R into **sub-rectangles** R_k, where $k = (k_1, k_2, \cdots, k_n) \in \mathbb{R}^n$ with $1 \leq k_j \leq \ell_j$ for each index j, and where

$$R_k = [x_{1,k_1-1}, x_{1,k_1}] \times [x_{2,k_2-1}, x_{2,k_2}] \times \cdots \times [x_{n,k_n-1}, x_{n,k_n}].$$

If Y is another partition, given by $Y_j = \{y_{j,0}, \cdots, y_{j,m_j}\}$, then we say that Y is **finer** than X (or that X is **coarser** than Y) when $X_j \subseteq Y_j$ for each index j.

8.2 Example: Note that a 1-dimensional rectangle in $\mathbb{R}^1$ is a line segment and its size is its length, a 2-dimensional rectangle in $\mathbb{R}^2$ is a rectangle and its size is its area, and a 3-dimensional rectangle in $\mathbb{R}^3$ is a rectangular box and its size is its volume.

8.3 Note: When R is a rectangle in $\mathbb{R}^n$ and X and Y are any two partitions of R, the partition Z given by $Z_j = X_j \cup Y_j$ is finer that both X and Y.

8.4 Note: When R is a rectangle in $\mathbb{R}^n$ and X is a partition given by $X_j = \{x_{j,0}, \cdots, x_{j,\ell_j}\}$, then letting $K = K(X) = \big\{ k \in \mathbb{Z}^n \,\big|\, 1 \leq k_j \leq \ell_j \text{ for all } j \big\}$, we have

$$\begin{aligned}
\sum_{k \in K} |R_k| &= \sum_{1 \leq k_1 \leq \ell_1} \sum_{1 \leq k_2 \leq \ell_2} \cdots \sum_{1 \leq k_n \leq \ell_n} \prod_{j=1}^{n} (x_{j,k_j} - x_{j,k_j-1}) \\
&= \prod_{j=1}^{n} \sum_{1 \leq k_j \leq \ell_j} (x_{j,k_j} - x_{j,k_j-1}) = \prod_{j=1}^{n} (x_{j,\ell_j} - x_{j,0}) \\
&= \prod_{j=1}^{n} (b_j - a_j) = |R|.
\end{aligned}$$

8.5 Definition: Let $A \subseteq \mathbb{R}^n$ be bounded. For a partition X of a rectangle R with $A \subseteq R$, we define the **upper** (or **outer**) **volume estimate of** A **with respect to** X, and the **lower** (or **inner**) **volume estimate of** A **with respect to** X, to be

$$U(A,X) = \sum_{R_k \cap \overline{A} \neq \emptyset} |R_k| = \sum_{k\in I} |R_k| \quad \text{and} \quad L(A,X) = \sum_{R_k \subseteq A^o} |R_k| = \sum_{k\in J} |R_k|$$

where $I = I(A,X) = \left\{k \in K \,|\, R_k \cap \overline{A} \neq \emptyset\right\}$ and $J = J(A,X) = \left\{k \in K \,\middle|\, R_k \subseteq A^o\right\}$ with $K = K(X) = \left\{k \in \mathbb{Z}^n \,\middle|\, 1 \leq k_j \leq \ell_j \text{ for each } j\right\}$.

8.6 Theorem: *(Basic Properties of Upper and Lower Volume Estimates) Let $A \subseteq \mathbb{R}^n$ be bounded, let R be a rectangle in $\mathbb{R}^n$ with $A \subseteq R$, and let X and Y be partitions of R.*
(1) If Y is finer than X then $0 \leq L(A,X) \leq L(A,Y) \leq U(A,Y) \leq U(A,X) \leq |R|$.
(2) $0 \leq L(A,X) \leq U(A,Y) \leq |R|$.
(3) $U(A,X) - L(A,X) = U(\partial A, X)$.

Proof: To prove Part 1, suppose that Y is finer than X. Note that each of the sub-rectangles R_k for the partition X is itself further partitioned into smaller sub-rectangles which are sub-rectangles for the partition Y, and denote these smaller sub-rectangles by $S_{k,1}, \cdots, S_{k,m_k}$. Then we have

$$U(A,X) = \sum_{k\in I} |R_k| \ \text{ and } \ U(A,Y) = \sum_{k\in I}\sum_{j\in J_k} |S_{k,j}|$$

where I is the set of $k \in K(X)$ such that $R_k \cap \overline{A} \neq \emptyset$ and J_k is the set of $j \in \{1, 2, \cdots, m_j\}$ such that $S_{k,j} \cap \overline{A} \neq \emptyset$. By Note 8.4, we have $\sum_{j=1}^{m_k} |S_{k,j}| = |R_k|$, and so

$$U(A,Y) = \sum_{k\in I}\sum_{j\in J_k} |S_{k,j}| \leq \sum_{k\in I}\sum_{j=1}^{m_j} |S_{k,j}| = \sum_{k\in I} |R_k| = U(A,X).$$

and also $U(A,X) = \sum_{k\in I} |R_k| \leq \sum_{k\in K(X)} |R_k| = |R|$. Thus we have $U(A,Y) \leq U(A,X) \leq |R|$. The proof that $L(A,X) \leq L(A,Y)$ is similar, and it is clear that $0 \leq L(A,X)$ and easy to see that $L(A,Y) \leq U(A,Y)$.

Note that Part 2 follows from Part 1 because, given any partitions X and Y for R, we can choose a partition Z which is finer than both X and Y, and then we have

$$0 \leq L(A,X) \leq L(A,Z) \leq U(A,Z) \leq U(A,Y) \leq |R|.$$

Finally, to prove Part 3, note that

$$U(A,X) - L(A,X) = \sum_{k\in L} |R_k| \ \text{ and } \ U(\partial A, X) = \sum_{k\in M} |R_k|$$

where L is the set of indices $k \in K(X)$ such that $R_k \cap \overline{A} \neq \emptyset$ and $R_k \not\subseteq A^o$, and M is the set of indices $k \in K(X)$ such that $R_k \cap \partial A \neq \emptyset$ (since ∂A is closed so that $\overline{\partial A} = \partial A$). We shall show that $K = M$. When $A = \emptyset$ we have $K = M = \emptyset$, so suppose $A \neq \emptyset$. If $k \in L$, that is if $R_k \cap \overline{A} \neq \emptyset$ and $R_k \not\subseteq A^o$ then we must have $R_k \cap \partial A \neq \emptyset$ because R_k is connected (indeed, if we had $R_k \cap \partial A = \emptyset$ then R_k would be separated by the disjoint nonempty open sets A^o and $\overline{A}^c$: note that we have $A^o \neq \emptyset$ because $R_k \cap \overline{A} \neq \emptyset$, and we have $\overline{A}^c \neq \emptyset$ because $R_k \not\subseteq A^o$) and hence $L \subseteq M$. If $k \in M$, that is if $R_k \cap \partial A \neq \emptyset$ then, since $\partial A \subseteq \overline{A}$ we have $R_k \cap \overline{A} \neq \emptyset$, and since A^o and ∂A are disjoint we have $R_k \not\subseteq A^o$, and hence $k \in M$. Thus $K = M$, as required.

8.7 Definition: Let $A \subseteq \mathbb{R}^n$ be bounded. We define the **upper** (or **outer**) **volume** (or **Jordan content**), and the **lower** (or **inner**) **volume** (or **Jordan content**), of A to be

$$U(A) = \inf\big\{U(A,X)\,\big|\,X \text{ is a partition of some rectangle } R \text{ with } A \subseteq R\big\}$$
$$L(A) = \sup\big\{L(A,X)\,\big|\,X \text{ is a partition of some rectangle } R \text{ with } A \subseteq R\big\}.$$

8.8 Theorem: *(Basic Properties of Upper and Lower Volumes) Let $A \subseteq \mathbb{R}^n$ be bounded.*
(1) If R is any rectangle with $A \subseteq R$ then $U(A) = \inf\big\{U(A,X)\,\big|\,X$ is a partition of $R\big\}$.
(2) $U(A) - L(A) = U(\partial A)$.

Proof: Given a rectangle R with $A \subseteq R$, let $U_R(A) = \inf\big\{U(A,X)\,\big|\,X$ is a partition of $R\big\}$. To prove Part 1, it suffices to show that for any two rectangles R, S in $\mathbb{R}^n$ which contain A, we have $U_R(A) = U_S(A)$. Let R and S be rectangles in $\mathbb{R}^n$ which contain A, say $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$ and $S = [c_1, d_1] \times \cdots \times [c_n, d_n]$.

Suppose first that $R \subseteq S$ with $c_j < a_j$ and $b_j < d_j$. Given any partition Y of S, we can extend Y to a finer partition Z of S by adding the endpoints of R, that is by letting $Z_j = Y_j \cup \{a_j, b_j\}$, and then we can restrict Z to a partition X of R as follows: if, for a fixed index j, we have $Z_j = \{z_0, \cdots, z_k, \cdots, z_\ell, \cdots, z_m\}$ with $z_0 = c_j$, $z_k = a_j$, $z_\ell = b_j$ and $z_m = d_j$, then we take $X_j = \{z_k, \cdots, z_\ell\}$. Then we have $U(A,X) \le U(A,Z) \le U(A,Y)$. Since for every partition Y of S there exists a corresponding partition X of R for which $U(A,X) \le U(A,Y)$, it follows that

$$\inf\big\{U(A,X)\,\big|\,X \text{ is a partition of } R\big\} \le \inf\big\{U(A,Y)\,\big|\,Y \text{ is a partition of } S\big\},$$

that is $U_R(A) \le U_S(A)$. Now let $\epsilon > 0$ and suppose that we are given a partition X of R. Choose s_j and t_j with $c_j < s_j < a_j$ and $b_j < t_j < b_j$ so that for the rectangle $T = [s_1, t_1] \times \cdots \times [s_n, t_n]$ we have $|T| - |R| \le \epsilon$. Extend the partition X of R to the partition Y of S by adding the endpoints of S and T, that is by letting $Y_j = X_j \cup \{c_j, s_j, t_j, d_j\}$. Note that the sub-rectangles of S which intersect with $\overline{A}$ include all of the sub-rectangles of R which intersect with $\overline{A}$ together with some of the sub-rectangles which lie in T but not R, and so we have $U(A,Y) \le U(A,X) + |T| - |R| \le U(A,X) + \epsilon$. Since for each partition X of R there is a corresponding partition Y of S for which $U(A,Y) \le U(A,X) + \epsilon$, it follows that

$$\inf\big\{U(A,Y)\,\big|\,Y \text{ is a partition of } S\big\} \le \inf\big\{U(A,X)\,\big|\,X \text{ is a partition of } R\big\} + \epsilon,$$

that is $U_S(A) \le U_R(A) + \epsilon$, and since $\epsilon > 0$ was arbitrary, it follows that $U_S(A) \le U_R(A)$. Thus we have proven that $U_R(A) = U_S(A)$ in the case that $R \subseteq S$ with $c_j < a_j < b_j < d_j$.

In the general case that $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$ and $S = [c_1, d_1] \times \cdots \times [c_n, d_n]$ are any rectangles which both contain A, we can choose a rectangle $T = [s_1, t_1] \times \cdots \times [s_n, t_n]$ with $s_j < \min\{a_j, c_j\}$ and $t_j > \max\{b_j, d_j\}$, and then we can apply the result of the above paragraph to obtain $U_R(A) = U_T(A) = U_S(A)$, as required, proving Part 1.

Let us prove Part 2. Given any partition X of any rectangle R containing A, we have $U(A) - L(A) \le U(A,X) - L(A,X) = U(\partial A, X)$, and hence (by taking the infemum on both sides) $U(A) - L(A) \le U(\partial A)$. It remains to show that $U(A) - L(A) \ge U(\partial A)$. Let $\epsilon > 0$. Choose a rectangle R containing A, and choose a partition X of R such that $L(A) - \epsilon < L(A,X) \le L(A)$. By Part 1, we can choose a partition Y of the same rectangle R such that $U(A) \le U(A,Y) < U(A) + \epsilon$. Let Z be a partition of R which is finer than both X and Y. Then we have $L(A) - \epsilon < L(A,X) \le L(A,Z)$ and $U(A,Z) \le U(A,Y) < U(A) + \epsilon$ and hence $U(\partial A) \le U(\partial A, Z) = U(A,Z) - L(A,Z) < U(A) - L(A) + 2\epsilon$. Since $\epsilon > 0$ was arbitrary, we have $U(\partial A) \le U(A) - L(A)$, as required.

8.9 Theorem: *For bounded sets $A, B \subseteq \mathbb{R}^n$, we have $U(A \cup B) \leq U(A) + U(B)$.*

Proof: First we note that for any sets $A, B \subseteq \mathbb{R}^n$ we have $\overline{A \cup B} = \overline{A} \cup \overline{B}$: Indeed, since $A \subseteq A \cup B$ and $B \subseteq A \cup B$ we have $\overline{A} \subseteq \overline{A \cup B}$ and $\overline{B} \subseteq \overline{A \cup B}$ so that $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. On the other hand, since $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$, we have $A \cup B \subseteq \overline{A} \cup \overline{B}$ and so, since $\overline{A} \cup \overline{B}$ is closed, and contains $A \cup B$, it follows that $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

Let $A, B \subseteq \mathbb{R}^n$ be bounded. Let R be a rectangle which contains $A \cup B$. Let $\epsilon > 0$. Choose a partition X of R so that $U(A) \leq U(A, X) + \frac{\epsilon}{2}$ and $U(B) \leq U(B, X) \leq \frac{\epsilon}{2}$ (we can do this by Part 1 of Theorems 8.8 and 8.6: let Y be a partition of R such that $U(A) \leq U(A, Y) + \frac{\epsilon}{2}$ let Z be a partition of R such that $U(B) \leq U(B, Z) + \frac{\epsilon}{2}$, then let X be a partition finer than both Y and Z). Let $K = K(X)$, let $I(A \cup B) = I(A \cup B, X)$, $I(A) = I(A, X)$ and $I(B) = I(B, X)$, as in Definition 8.5. Since $\overline{A \cup B} = \overline{A} \cup \overline{B}$, for each index $k \in K$ we have

$$k \in I(A \cup B) \iff R_k \cap \overline{A \cup B} \neq \emptyset \iff (R_k \cap \overline{A}) \cup (R_k \cap \overline{B}) \neq \emptyset \iff \big(k \in I(A) \text{ or } k \in I(B)\big),$$

$$U(A \cup B, X) = \sum_{k \in I(A \cup B)} |R_k| \leq \sum_{k \in I(A)} |R_k| + \sum_{k \in I(B)} |R_k| = U(A, X) + U(B, X) \leq U(A) + U(B) + \epsilon.$$

Since $U(A \cup B, X) \leq U(A) + U(B) + \epsilon$ for all partitions X of R, it follows (from Part 1 of Theorem 8.8) that $U(A \cup B) \leq U(A) + U(B) + \epsilon$, and since $\epsilon > 0$ was arbitrary, it follows that $U(A \cup B) \leq U(A) + U(B)$, as required.

8.10 Definition: Let $A \subseteq \mathbb{R}^n$ be bounded. We say that A **has well-defined volume** (or **Jordan content**), or that A is **Jordan measurable**, or that A is a **Jordan region**, when $U(A) = L(A)$, or equivalently (by Part 2 of Theorem 8.8) when $U(\partial A) = 0$. In this case, we define the (n-dimensional) **volume** of A (or the **Jordan content**) of A to be

$$\mathrm{Vol}(A) = U(A) = L(A).$$

8.11 Theorem: *Every rectangle R in $\mathbb{R}^n$ is Jordan measurable with* $\mathrm{Vol}(R) = |R|$.

Proof: Let $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$ be a ractangle in $\mathbb{R}^n$. By Note 8.4, we have $U(R, X) = |R|$ for every partition X of R, so by Part 1 of Theorem 8.8, it follows that $U(R) = |R|$. By Part 2 of Theorem 8.8, we have $U(R) - L(R) = U(\partial R) \geq 0$ so that $L(R) \leq U(R)$. Let $\epsilon > 0$. Choose a rectangle S of the form $S = [c_1, d_1] \times \cdots \times [c_n, d_n]$ with $a_1 < c_1$ and $d_1 < b_1$ (so that $S \subseteq R^o$) such that $|R| - |S| < \epsilon$. Let X be the partition of R given by $X_j = \{a_j, c_j, d_j, b_j\}$. Since S is a sub-rectangle for this partition with $S \subseteq R^o$ we have $L(R, X) \geq |S|$, and so $L(R) \geq L(R, X) \geq |S| > |R| - \epsilon$. Since $\epsilon > 0$ was arbitrary, it follows that $L(R) \geq |R|$. Thus we have $L(R) = |R| = U(R)$.

8.12 Theorem: *(Properties of Jordan Content) Let $A, B \subseteq \mathbb{R}^n$ be Jordan measurable.*
(1) If $A \subseteq B$ then $\mathrm{Vol}(A) \leq \mathrm{Vol}(B)$.
(2) A^o and $\overline{A}$ are Jordan measurable with $\mathrm{Vol}(A^o) = \mathrm{Vol}(A) = \mathrm{Vol}(\overline{A})$.
(3) $A \cup B$, $A \cap B$ and $A \setminus B$ are Jordan measurable with $\mathrm{Vol}(A \setminus B) = \mathrm{Vol}(A) - \mathrm{Vol}(A \cap B)$ *and* $\mathrm{Vol}(A \cup B) = \mathrm{Vol}(A) + \mathrm{Vol}(B) - \mathrm{Vol}(A \cap B)$. *If $A \cap B = \emptyset$ then* $\mathrm{Vol}(A \cup B) = \mathrm{Vol}(A) + \mathrm{Vol}(B)$.

Proof: To prove Part 1, suppose that $A \subseteq B$. Let R be a rectangle containing B and let X be a partition of R into the sub-rectangles R_k with $k \in K(X)$. Since $A \subseteq B$, we have $\overline{A} \subseteq \overline{B}$, so for $k \in K(X)$, if $R_k \cap \overline{A} \neq \emptyset$ then $R_k \cap \overline{B} \neq \emptyset$. This shows that $I(A, X) \subseteq I(B, X)$ and hence $U(A, X) = \sum_{k \in I(A, X)} |R_k| \leq \sum_{k \in I(B, X)} |R_k| = U(B, X)$. Since $U(A, X) \leq U(B, X)$ for every partition X of R, we have $U(A) \leq U(B)$ (by Part 1 of Theorem 8.8). Since A and B are measurable, this means that $\mathrm{Vol}(A) \leq \mathrm{Vol}(B)$, as required.

Let us prove Part 2. Since A^o is open we have $(A^o)^o = A^o$, and since $A^o \subseteq A$ we have $\overline{A^o} \subseteq \overline{A}$, and hence $\partial(A^o) = \overline{A^o} \setminus (A^o)^o = \overline{A^o} \setminus A^o \subseteq \overline{A} \setminus A^o = \partial A$. Since $\partial A^o \subseteq \partial A$ we have $U(\partial A^o) \leq U(\partial A)$ (by Part 1), and since A is measurable we have $U(\partial A) = 0$. Thus $U(\partial A^o) = 0$ so that A^o is Jordan measurable. Similarly, we have $\overline{\overline{A}} = \overline{A}$ and $A^o \subseteq \overline{A}^o$ so that $\partial \overline{A} = \overline{\overline{A}} \setminus \overline{A}^o = \overline{A} \setminus \overline{A}^0 \subseteq \overline{A} \setminus A^o = \partial A$ and hence $U(\partial \overline{A}) \leq U(\partial A) = 0$ so that $\overline{A}$ is Jordan measurable. Now let R be a rectangle containing A and let X be a partition of R. From the definition of $U(A, X)$ it is immediate that $U(A, X) = U(\overline{A}, X)$, and from the definition of $L(A, X)$ it is immediate that $L(A, X) = L(A^o, X)$. Since this holds for all partitions X of R, we have $U(A) = U(\overline{A})$ and $L(A) = L(A^o)$. Since A is measurable, this gives $L(A^o) = L(A) = U(A) = U(\overline{A})$, and since A^o and $\overline{A}$ are measurable, this gives $\mathrm{Vol}(A^o) = \mathrm{Vol}(A) = \mathrm{Vol}(\overline{A})$, as required.

We move on to the proof of Part 3. To prove that $A \cup B$ is Jordan measurable, we note that $\partial(A \cup B) \subseteq \partial A \cup \partial B$: indeed, recall (as shown in the proof of Theorem 8.9) that $\overline{A \cup B} = \overline{A} \cup \overline{B}$. Also note that since $A \subseteq A \cup B$ and $B \subseteq A \cup B$ we have $A^o \subseteq (A \cup B)^o$ and $B^o \subseteq (A \cup B)^o$ so that $A^o \cup B^o \subseteq (A \cup B)^o$. Thus

$$\begin{aligned} x \in \partial(A \cup B) &\implies x \in \overline{A \cup B} \text{ and } x \notin (A \cup B)^o \\ &\implies x \in \overline{A} \cup \overline{B} \text{ and } x \notin A^o \cup B^o \\ &\implies \big(x \in \overline{A} \text{ and } x \notin A^o\big) \text{ and } \big(x \in \overline{B} \text{ and } x \notin B^o\big) \\ &\implies x \in \partial A \cup \partial B. \end{aligned}$$

Since $\partial(A \cup B) \subseteq \partial A + \partial B$, Theorem 8.9 gives $U(\partial(A \cup B)) \leq U(\partial A) + U(\partial B)$. Since A and B are Jordan measurable so that $U(\partial A) = 0$ and $U(\partial B) = 0$, we also have $U(\partial(A \cup B)) = 0$ so that $A \cup B$ is Jordan measurable. We can prove that $A \cap B$ and $A \setminus B$ are measuable in the same way, by showing that $\partial(A \cap B) \subseteq \partial A \cup \partial B$ and $\partial(A \setminus B) \subseteq \partial A \cup \partial B$, and we leave this as an exercise.

It remains to prove the various volume formulas. First, suppose that $A \cap B = \emptyset$. We know, from Theorem 8.9 that $U(A \cap B) \leq U(A) + U(B)$. Let R be a rectangle which contains $A \cup B$, and let X be a partition of R such that $L(A, X) \geq L(A) - \frac{\epsilon}{2}$ and $L(B, X) \geq L(B) - \frac{\epsilon}{2}$. Since $A^o \subseteq A \subseteq A \cup B \subseteq \overline{A \cup B}$, it follows that if $k \in J(A^o, X)$, that is if $R_k \subseteq A^0$, then we have $R_k \subseteq \overline{A \cup B}$ so that $R_k \cap \overline{A \cup B} \neq \emptyset$, that is $k \in I(A \cap B, X)$, so we have $J(A, X) \subseteq I(A \cup B, X)$. Similarly, since $B^o \subseteq \overline{A \cup B}$, we have $J(B, X) \subseteq I(A \cup B, X)$. Also note that since $A \cap B = \emptyset$, we also have $A^o \cap B^o = \emptyset$, so it is not possible to have both $R_k \subseteq A^o$ and $R_k \subseteq B^o$, and it follows that $J(A, X) \cap J(B, X) = \emptyset$. Thus

$$U(A \cup B, X) = \sum_{k \in I(A \cap B, X)} |R_k| \geq \sum_{k \in J(A,X)} |R_k| + \sum_{k \in J(B,X)} |R_k| = L(A, X) + L(B, X) \geq L(A) + L(B) - \epsilon.$$

Since $U(A \cup B, X) \geq L(A) + L(B) - \epsilon$ for all partitions X of R, and since $\epsilon > 0$ was arbitrary, we have $U(A \cup B) \geq L(A) + L(B)$. Together with Theorem 8.9, this gives

$$L(A) + L(B) \leq U(A \cup B) \leq U(A) + U(B).$$

Since $L(A) = U(A) = \mathrm{Vol}(A)$ and $L(B) = U(B) = \mathrm{Vol}(B)$ and $U(A \cup B) = \mathrm{Vol}(A \cup B)$, we have proven that, if $A \cap B = \emptyset$ then $\mathrm{Vol}(A \cup B) = \mathrm{Vol}(A) + \mathrm{Vol}(B)$.

Finally, we note that the other two formulas (which apply whether or not A and B are disjoint), follow from the special case of disjoint sets: Indeed, the set A is the disjoint union $A = (A \setminus B) \cup (A \cap B)$, so we have $\mathrm{Vol}(A) = \mathrm{Vol}(A \setminus B) + \mathrm{Vol}(A \cap B)$, and $A \cup B$ is the disjoint union $A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap H)$ so that $\mathrm{Vol}(A \cup B) = \mathrm{Vol}(A \setminus B) + \mathrm{Vol}(B \setminus A) + \mathrm{Vol}(A \cap B) = \mathrm{Vol}(A) + \mathrm{Vol}(B) - \mathrm{Vol}(A \cap B)$.

8.13 Definition: A **cube** in $\mathbb{R}^n$ is a rectangle $Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$ in $\mathbb{R}^n$ with equal side lengths, that is with $b_k - a_k = b_\ell - a_\ell$ for all $k \neq \ell$.

8.14 Theorem: *(Alternate Characterizations of Outer Jordan Content) Let $A \subseteq \mathbb{R}^n$ be bounded. Then*

$$\begin{aligned} U(A) &= \inf \left\{ \sum_{j=1}^m |R_j| \,\middle|\, R_1, R_2, \cdots, R_m \textit{ are rectangles } A \subseteq \bigcup_{j=1}^m R_j \right\} \\ &= \inf \left\{ \sum_{j=1}^m |Q_j| \,\middle|\, Q_1, Q_2, \cdots, Q_m \textit{ are cubes of equal size with } A \subseteq \bigcup_{j=1}^m Q_j \right\}. \end{aligned}$$

Proof: Let

$$\mathcal{R} = \left\{ \sum_{R_k \cap \overline{A} \neq \emptyset}^m |R_k| \,\middle|\, X \text{ is a partition of some rectangle } R \text{ with } A \subseteq R \right\},$$

$$\mathcal{S} = \left\{ \sum_{j=1}^m |R_j| \,\middle|\, R_1, R_2, \cdots, R_m \text{ are rectangles with } A \subseteq \bigcup_{j=1}^m R_j \right\} \text{, and}$$

$$\mathcal{T} = \left\{ \sum_{j=1}^m |Q_j| \,\middle|\, Q_1, Q_2, \cdots, Q_m \text{ are squares of equal size with } A \subseteq \bigcup_{j=1}^m Q_j \right\}.$$

and note that $U(A) = \inf \mathcal{R}$. We leave the proof that $U(A) = \inf \mathcal{S}$ as an exercise, and we prove that $U(A) = \inf \mathcal{T}$. When $Q_1, \cdots, Q_m$ are cubes of equal size with $A \subseteq \bigcup_{k=1}^m Q_k$, we know that $U(A) \leq \sum_{k=1}^m |Q_k|$ by Theorem 8.9, and hence $U(A) \leq \inf \mathcal{S}$. It remains to show that $\inf \mathcal{S} \leq U(A)$.

Let $\epsilon > 0$. Choose a rectangle R with $A \subseteq R$, and choose a partition X of R into sub-rectangles R_k such that $U(A, X) \leq U(A) + \frac{\epsilon}{2}$. Let $k_1, \cdots, k_m$ be the values of k for which $R_k \cap \overline{A} \neq \emptyset$, so we have $\overline{A} \subseteq \bigcup_{i=1}^m R_{k_i}$ and $\sum_{i=1}^m |R_{k_i}| = U(A, X) \leq U(A) + \frac{\epsilon}{2}$. For each index i, choose a rectangle S_i with $R_{k_i} \subseteq S_i$ such that the endpoints of all the component intervals of all the rectangles S_i are rational and $\sum_{i=1}^m |S_i| \leq \sum_{i=1}^m |R_{k_i}| + \frac{\epsilon}{2}$. Let d be a common denominator of all the endpoints of all the rectangles S_i, and partition each rectangle S_i into cubes $Q_{i,1}, Q_{i,2}, \cdots, Q_{i,\ell_i}$ all with sides of length $\frac{1}{d}$. Then we have $A \subseteq \bigcup_{i=1}^m S_i = \bigcup_{i=1}^m \bigcup_{j=1}^{\ell_i} Q_{i,j}$ and

$$\sum_{i=1}^m \sum_{j=1}^{\ell_i} |Q_{i,j}| = \sum_{i=1}^m |S_i| \leq \textstyle\sum_{i=1}^m |R_{k_i}| + \frac{\epsilon}{2} \leq U(A) + \epsilon.$$

Thus $\inf \mathcal{S} \leq U(A) + \epsilon$. Since $\epsilon > 0$ was arbitrary, we have $\inf \mathcal{S} \leq U(a)$, as required.

8.15 Definition: For a map $g : A \subseteq \mathbb{R}^n \to B \subseteq \mathbb{R}^m$, we say that g is **Lipschitz continuous** on A when there is a constant $c \geq 0$ such that $|g(x) - g(y)| \leq c|x - y|$ for all $x, y \in A$, and we say that g is **open** when $g(U)$ is open in B for every open set U in A.

8.16 Theorem: *Let $A \subseteq \mathbb{R}^n$ be bounded and let $g : A \to \mathbb{R}^n$ be Lipschitz continuous.*

(1) If $U(A) = 0$ and g is Lipschitz continuous then $U(g(A)) = 0$.
(2) If A is Jordan measurable and g is open then $g(A)$ is Jordan measurable.

Proof: The proof is left as an exercise.

8.17 Definition: Let $A \subseteq \mathbb{R}^n$ be a Jordan region and let $f : A \to \mathbb{R}$ be a bounded function. Let X be a partition of a rectangle R in $\mathbb{R}^n$ which contains A, and let R_k, $k \in K$ be the sub-rectangles. Extend f to a function $g : R \to \mathbb{R}$ by defining $g(x) = f(x)$ when $x \in A$ and $g(x) = 0$ when $x \in R \setminus A$. The **upper Riemann sum** of f on A for the partition X and the **lower Riemann sum** of f on A for X are given by

$$U(f,X) = \sum_{k\in K} M_k\,|R_k| \quad\text{and}\quad L(f,X) = \sum_{k\in K} m_k\,|R_k|$$

where $M_k = \sup\big\{g(x)\,|\,x \in R_k\big\}$ and $m_k = \inf\{f(x)\,|\,x \in R_k\}$. The **upper integral** of f on A and the **lower integral** of f on A are given by

$$U(f) = \inf\big\{U(f,X)\,\big|\, X \text{ is a partition of some rectangle } R \text{ with } A \subseteq R\big\}$$
$$L(f) = \sup\big\{L(f,X)\,\big|\, X \text{ is a partition of some rectangle } R \text{ with } A \subseteq R\big\}.$$

We say that f is (Riemann) **integrable** on A when $U(f) = L(f)$ and, in this case, we define the (Riemann) **integral** of f on A to be

$$\int_A f = \int_A f(x)\,dV = \int_A f(x_1,\cdots,x_n)\,dx_1dx_2\cdots dx_n = U(f) = L(f).$$

8.18 Theorem: *(Properties of Upper and Lower Riemann Sums) Let $A \subseteq \mathbb{R}^n$ be a Jordan region, let $f : A \to \mathbb{R}$ be a bounded function, let R be a rectangle which contains A, and let X and Y be two partitions of R.*
(1) If Y is finer than X then $L(f,X) \le L(f,Y) \le U(f,Y) \le U(f,X)$.
(2) We have $L(f,X) \le U(f,Y)$.

Proof: Let $g : R \to \mathbb{R}$ be the extension of f by zero. When $M_k = \sup\{g(x)\,|\,x\in R_k\}$ and $m_k = \inf\{g(x)\,|\,x\in R_k\}$, we have $m_k \le M_k$ for all $k \in K = K(X)$ so that

$$L(f,X) = \sum_{k\in K} m_k|R_k| \le \sum_{k\in K} M_k|R_k| = U(f,X).$$

Similarly, we have $L(f,Y) \le U(f,Y)$.

Suppose that Y is finer than X. Note that each of the sub-rectangles R_k for the partition X is itself further partitioned into smaller sub-rectangles which are sub-rectangles for the partition Y, and denote these smaller sub-rectangles by $S_{k,1},\cdots,S_{k,m_k}$. Note that $|R_k| = \sum_{j=1}^{m_k}|S_{k,j}|$ by Note 8.4. Let $M_k = \sup\{g(x)|\,x\in R_k\}$ and $N_{k,j} = \sup\{g(x)|\,x\in S_{k,j}\}$. Since $R_k = \bigcup_{j=1}^{m_k} S_{k,j}$, we have $M_k = \max\{N_{k,j}|1\le j\le m_k\}$ and hence

$$U(f,X) = \sum_{k\in K} M_k|R_k| = \sum_{k\in K}\sum_{j=1}^{m_k} M_k|S_{k,j}| \ge \sum_{k\in K}\sum_{j=1}^{m_k} N_{k,j}|S_{k,j}| = U(f,Y).$$

A similar argument shows that $L(f,X) \le L(f,Y)$. This completes the proof of Part 1.

Part 2 follows from Part 1. Indeed, given any partitions X and Y of R, we can choose a partition Z which is finer than both X and Y, and then we have

$$L(f,X) \le L(f,Z) \le U(f,Z) \le U(f,Y).$$

8.19 Theorem: *(Properties of Upper and Lower Integrals) Let $A \subseteq \mathbb{R}^n$ be a Jordan region, and let $f : A \to \mathbb{R}$ be a bounded function.*
(1) If R is any rectangle with $A \subseteq \mathbb{R}^n$ then $U(f) = \inf\big\{U(f,X)\,\big|\,X$ is a partition of $R\big\}$ and $L(f) = \sup\big\{L(f,X)\,\big|\,X$ is a partition of $R\big\}$.
(2) We have $L(f) \le U(f)$.

Proof: To prove Part 1, imitate the proof of Part 1 of Theorem 8.8. Part 2 follows from Part 1 of this theorem together with Part 2 of the previous theorem.

8.20 Theorem: *(Characterization of Integrability) Let $A \subseteq \mathbb{R}^n$ be a Jordan region, and let $f : A \to \mathbb{R}$ be a bounded function. Then f is integrable on A if and only if for every $\epsilon > 0$ there exits a partition X of a rectangle R with $A \subseteq R$ such that $U(f, X) - L(f, X) < \epsilon$.*

Proof: Suppose that f is integrable on A, so we have $U(f) = L(f)$. Let R be a rectangle with $A \subseteq R$. By Part 1 of Theorem 8.19, we can choose a partition Y of R such that $U(f, Y) < U(f) + \frac{\epsilon}{2}$, and we can choose a partition Z of R such that $L(f, Z) > L(f) - \frac{\epsilon}{2}$. Let X be a partition of R which is finer than both Y and Z. By Part 1 of Theorem 8.18, we have $U(f, X) \leq U(f, Y)$ and $L(f, X) \geq L(f, Z)$, and hence

$$U(f,X) - L(f,X) \leq U(f,Y) - L(f,Z) < \big(U(f) + \tfrac{\epsilon}{2}\big) - \big(L(f) - \tfrac{\epsilon}{2}\big) = U(f) - L(f) + \epsilon = \epsilon.$$

Suppose, conversely, that for every $\epsilon > 0$ there exists a partition X of a rectangle R with $A \subseteq R$ such that $U(f, X) - L(f, X) < \epsilon$. Let $\epsilon > 0$. Choose R and X so that $U(f, X) - L(f, X) < \epsilon$. By the definition of $U(f)$ and $L(f)$, we have $U(f) \leq U(f, X)$ and $L(f) \geq L(f, X)$, and so $U(f) - L(f) \leq U(f, X) - L(f, X) < \epsilon$. Since $U(f) - L(f) < \epsilon$ for every $\epsilon > 0$, it follows that $U(f) \leq L(f)$. On the other hand, we have $U(f) \geq L(f)$ by Part 2 of Theorem 8.19. Thus $U(f) = L(f)$ so that f is integrable.

8.21 Theorem: *(Continuity and Integrability) Let $A \subseteq \mathbb{R}^n$ be a Jordan region, and let $f : A \to \mathbb{R}$ be a bounded function. If f is uniformly continuous on A, then f is integrable.*

Proof: Suppose that f is bounded and uniformly continuous on A. Choose a rectangle R with $A \subseteq R$ and $|R| > 0$. Let $\epsilon > 0$. Since f is bounded, we can choose $M > 0$ so that $|f(x)| \leq M$ for all $x \in A$. Since f is uniformly continuous on A, we can choose $\delta > 0$ such that for all $x, y \in A$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \frac{\epsilon}{2|R|}$. Choose a partition X of R, into sub-rectangles R_k, which is fine enough so that firstly, we have $x, y \in R_k \implies |x - y| < \delta$ and, secondly, we have $U(\partial A, X) = \sum_{R_k \cap \partial A \neq \emptyset} |R_k| < \frac{\epsilon}{2M}$ (we can do this since $U(\partial A) = 0$). Since $\overline{A}$ is the disjoint union $\overline{A} = A^o \cup \partial A$, the rectangles R_k come in three varieties: $R_k \cap \overline{A} = \emptyset$, $R_k \cap \partial A \neq \emptyset$ or $R_k \subseteq A^o$. Let g be the extension of f by zero to R, and write $M_k = \sup\{g(x) | x \in R_k\}$ and $m_k = \inf\{g(x) | x \in R_k\}$. When $R_k \cap \overline{A} = \emptyset$, we have $g(x) = 0$ for all $x \in R_k$, and so

$$\sum_{R_k \cap \overline{A} = \emptyset} (M_k - m_k)|R_k| = 0.$$

When $R_k \cap \partial A \neq \emptyset$ we have $|g(x)| \leq M$ for all $x \in R_k$ so that

$$\sum_{R_k \cap \partial A \neq \emptyset} (M_k - m_k)|R_k| \leq 2M \sum_{R_k \cap \partial A \neq \emptyset} |R_k| < \tfrac{\epsilon}{2}.$$

When $R_k \subseteq A^o$, for all $x, y \in R_k$ we have $x, y \in A$ with $|x - y| < \delta$ so that $|g(x) - g(y)| = |f(x) - f(y)| < \frac{\epsilon}{2|R|}$, and hence $M_k - m_k \leq \frac{\epsilon}{2|R|}$ so that

$$\sum_{R_k \subseteq A^o} (M_k - m_k)|R_k| \leq \tfrac{\epsilon}{2|R|} \sum_{R_k \subseteq A^o} |R_k| \leq \tfrac{\epsilon}{2}.$$

Thus

$$U(f,X) - L(f,X) = \sum_{R_k \cap \overline{A} = \emptyset} (M_k - m_k)|R_k| + \sum_{R_k \cap \partial A \neq \emptyset} (M_k - m_k)|R_k| + \sum_{R_k \subseteq A^o} (M_k - m_k)|R_k| < \epsilon.$$

Thus f is integrable, by Theorem 8.20.

8.22 Theorem: *(Integration and Volume) If $A \subseteq \mathbb{R}^n$ is a Jordan region then*

$$\mathrm{Vol}(A) = \int_A 1\, dV.$$

Proof: Suppose that A is Jordan measurable, so we have $U(A) = L(A) = \mathrm{Vol}(A)$. Let R be a rectangle with $A \subseteq R$. Let $f : A \to \mathbb{R}$ be the constant function $f(x) = 1$, and let $g : R \to \mathbb{R}$ be the extension of f by zero. Choose a partition X of R, with sub-rectangles R_k, such that $U(A, X) \leq U(A) - \epsilon = \mathrm{Vol}(A) - \epsilon$ and $L(A, X) \geq L(A) - \epsilon = \mathrm{Vol}(A) - \epsilon$. Let $M_k = \sup\{g(x) | x \in R_k\}$ and $m_k = \inf\{g(x) | x \in R_k\}$. When $R_k \cap \overline{A} = \emptyset$ we have $g(x) = 0$ for all $x \in R_k$ so that $M_k = 0$, and for all k we have $M_k \leq 1$, and so

$$U(f) \leq U(f, X) = \sum_{R_k \cap \overline{A} \neq \emptyset} M_k |R_k| \leq \sum_{R_k \cap \overline{A} \neq \emptyset} |R_k| = U(A, X) \leq \mathrm{Vol}(A) + \epsilon.$$

When $R_k \subseteq A^o$ we have $g(x) = 1$ for all $x \in R_k$ so that $m_k = 1$, and for all k we have $m_k \geq 0$, and so

$$L(f) \geq L(f, X) \geq \sum_{R_k \subseteq A^o} m_k |R_k| = \sum_{R_k \subseteq A^o} |R_k| = L(A, X) \geq \mathrm{Vol}(A) - \epsilon.$$

Since $\mathrm{Vol}(A) - \epsilon \leq L(f) \leq U(f) \leq \mathrm{Vol}(A) + \epsilon$ for every $\epsilon > 0$, we have $U(f) = L(f) = \mathrm{Vol}(A)$, which means that f is integrable on A with $\int_A 1 = \int_A f = \mathrm{Vol}(A)$, as required.

8.23 Theorem: *(Linearity) Let $A \subseteq \mathbb{R}^n$ be a Jordan region and let $f, g : A \to \mathbb{R}$ be integrable. Then $f + g$ is integrable, and cf is integrable for every $c \in \mathbb{R}$, and we have*

$$\int_A (f + g) = \int_A f + \int_A g \quad \text{and} \quad \int_A cf = c \int_A f.$$

Proof: The proof is left as an exercise.

8.24 Theorem: *(Decomposition) Let A and B be Jordan regions in $\mathbb{R}^n$ with $\mathrm{Vol}(A \cap B) = 0$, and let $f : A \cup B \to \mathbb{R}$ be bounded. Let $g : A \to \mathbb{R}$ be the restrictions of f to A and let $h : B \to \mathbb{R}$ be the restriction of f to B. Then f is integrable on $A \cup B$ if and only if g is integrable on A and h is integrable on B and, in this case, we have*

$$\int_{A \cup B} f = \int_A g + \int_B h.$$

Proof: The proof is left as an exercise.

8.25 Theorem: *(Comparison) Let A be a Jordan region in $\mathbb{R}^n$ and let $f, g : A \to \mathbb{R}$ be integrable. If $f(x) \leq g(x)$ for all $x \in A$ then $\int_A f \leq \int_A d$.*

Proof: The proof is left as an exercise.

8.26 Theorem: *(Absolute Value) Let A be a Jordan region in $\mathbb{R}^n$ and let $f : A \to \mathbb{R}$ be integrable. Then the function $|f|$ is integrable and $\left| \int_A f \right| \leq \int_A |f|$.*

Proof: The proof is left as an exercise.