Chapter 2. Topological Properties of Sets in Euclidean Space

2.1 Definition: For vectors $x, y \in \mathbb{R}^n$ we define the **dot product** of x and y to be

$$x \cdot y = y^T x = \sum_{i=1}^{n} x_i y_i \,.$$

2.2 Theorem: *(Properties of the Dot Product) For all $x, y, z \in \mathbb{R}^n$ and all $t \in \mathbb{R}$ we have*
(1) (Bilinearity) $(x+y)\cdot z = x\cdot z + y\cdot z\ ,\ \ (tx)\cdot y = t(x\cdot y)$
$x\cdot(y+z) = x\cdot y + x\cdot z\ ,\ \ x\cdot(ty) = t(x\cdot y)$,
(2) (Symmetry) $x\cdot y = y\cdot x$, *and*
(3) (Positive Definiteness) $x\cdot x \geq 0$ *with* $x\cdot x = 0$ *if and only if* $x = 0$.

Proof: The proof is left as an exercise.

2.3 Definition: For a vector $x \in \mathbb{R}^n$, we define the **norm** (or **length**) of x to be

$$|x| = \sqrt{x\cdot x} = \sqrt{\sum_{i=1}^{n} {x_i}^2}.$$

We say that x is a **unit vector** when $|x| = 1$.

2.4 Theorem: *(Properties of the Norm) Let $x, y \in \mathbb{R}^n$ and let $t \in \mathbb{R}$. Then*
(1) (Positive Definiteness) $|x| \geq 0$ *with* $|x| = 0$ *if and only if* $x = 0$,
(2) (Scaling) $|tx| = |t||x|$,
(3) $|x \pm y|^2 = |x|^2 \pm 2(x\cdot y) + |y|^2$.
(4) (The Polarization Identities) $x\cdot y = \frac{1}{2}\big(|x+y|^2 - |x|^2 - |y|^2\big) = \frac{1}{4}\big(|x+y|^2 - |x-y|^2\big)$,
(5) (The Cauchy-Schwarz Inequality) $|x\cdot y| \leq |x|\,|y|$ *with* $|x\cdot y| = |x|\,|y|$ *if and only if the set* $\{x, y\}$ *is linearly dependent, and*
(6) (The Triangle Inequality) $|x+y| \leq |x| + |y|$.

Proof: We leave the proofs of Parts (1), (2) and (3) as an exercise, and we note that (4) follows immediately from (3). To prove part (5), suppose first that $\{x, y\}$ is linearly dependent. Then one of x and y is a multiple of the other, say $y = tx$ with $t \in \mathbb{R}$. Then

$$|x\cdot y| = |x\cdot(tx)| = |t(x\cdot x)| = |t|\,|x|^2 = |x|\,|tx| = |x|\,|y|.$$

Suppose next that $\{x, y\}$ is linearly independent. Then for all $t \in \mathbb{R}$ we have $x + ty \neq 0$ and so

$$0 \neq |x+ty|^2 = (x+ty)\cdot(x+ty) = |x|^2 + 2t(x\cdot y) + t^2|y|^2.$$

Since the quadratic on the right is non-zero for all $t \in \mathbb{R}$, it follows that the discriminant of the quadratic must be negative, that is

$$4(x\cdot y)^2 - 4|x|^2|y|^2 < 0.$$

Thus $(x\cdot y)^2 < |x|^2|y|^2$ and hence $|x\cdot y| < |x|\,|y|$. This proves part (5).

Using part (5) note that

$$|x+y|^2 = |x|^2 + 2(x\cdot y) + |y|^2 \leq |x+y|^2 + 2|x\cdot y| + |y|^2 \leq |x|^2 + 2|x|\,|y| + |y|^2 = \big(|x| + |y|\big)^2$$

and so $|x+y| \leq |x| + |y|$, which proves part (6).

2.5 Definition: For points $a, b \in \mathbb{R}^n$, we define the **distance** between a and b to be

$$\text{dist}(a, b) = |b - a|.$$

2.6 Theorem: *(Properties of Distance) Let $a, b, c \in \mathbb{R}^n$. Then*
(1) (Positive Definiteness) $\text{dist}(a, b) \geq 0$ with $\text{dist}(a, b) = 0$ if and only if $a = b$,
(2) (Symmetry) $\text{dist}(a, b) = \text{dist}(b, a)$, and
(3) (The Triangle Inequality) $\text{dist}(a, c) \leq \text{dist}(a, b) + \text{dist}(b, c)$.

Proof: The proof is left as an exercise.

2.7 Definition: For nonzero vectors $0 \neq u, v \in \mathbb{R}^n$, we define the **angle between** u and v to be $\theta(u, v) = \cos^{-1} \frac{u \cdot v}{|u|\,|v|} \in [0, \pi]$. We say that u and v are **orthogonal** when $u \cdot v = 0$. As an exercise, determine (with proof) some properties of angles.

2.8 Definition: For $a \in \mathbb{R}^n$ and $0 < r \in \mathbb{R}$, the **sphere**, the **open ball**, the **closed ball**, and the (open) **punctured ball** in $\mathbb{R}^n$ centered at a of radius r are defined to be the sets

$$\begin{aligned} S(a, r) &= \big\{x \in \mathbb{R}^n \big| \text{dist}(x, a) = r\big\} = \big\{x \in \mathbb{R}^n \big| |a - x| = r\big\}, \\ B(a, r) &= \big\{x \in \mathbb{R}^n \big| \text{dist}(x, a) < r\big\} = \big\{x \in \mathbb{R}^n \big| |a - x| < r\big\}, \\ \overline{B}(a, r) &= \big\{x \in \mathbb{R}^n \big| \text{dist}(x, a) \leq r\big\} = \big\{x \in \mathbb{R}^n \big| |a - x| \leq r\big\}, \\ B^*(a, r) &= \big\{x \in \mathbb{R}^n \big| 0 < \text{dist}(x, a) < r\big\} = \big\{x \in \mathbb{R}^n \big| 0 < |a - x| < r\big\}. \end{aligned}$$

2.9 Definition: Let $A \subseteq \mathbb{R}^n$. We say that A is **bounded** when $A \subseteq B(a, r)$ for some $a \in \mathbb{R}^n$ and some $0 < r \in \mathbb{R}$. As an exercise, verify that A is bounded if and only if $A \subseteq B(0, r)$ for some $r > 0$.

2.10 Definition: For a set $A \subseteq \mathbb{R}^n$, we say that A is **open** (in $\mathbb{R}^n$) when for every $a \in A$ there exists $r > 0$ such that $B(a, r) \subseteq A$, and we say that A is **closed** (in $\mathbb{R}^n$) when its complement $A^c = \mathbb{R}^n \setminus A$ is open in $\mathbb{R}^n$.

2.11 Exercise: Show that open intervals in $\mathbb{R}$ are open in $\mathbb{R}$ and closed intervals in $\mathbb{R}$ are closed in $\mathbb{R}$.

2.12 Example: Show that for $a \in \mathbb{R}^n$ and $0 < r \in \mathbb{R}$, the set $B(a, r)$ is open and the set $\overline{B}(a, r)$ is closed.

Solution: Let $a \in \mathbb{R}^n$ and let $r > 0$. We claim that $B(a, r)$ is open. We need to show that for all $b \in B(a, r)$ there exists $s > 0$ such that $B(b, s) \subseteq B(a, r)$. Let $b \in B(a, r)$ and note that $|b - a| < r$. Let $s = r - |b - a|$ and note that $s > 0$. Let $x \in B(b, s)$, so we have $|x - b| < s$. Then, by the Triangle Inequality, we have

$$|x - a| = |x - b + b - a| \leq |x - b| + |b - a| < s + |b - a| = r$$

and so $x \in B(a, r)$. This shows that $B(b, s) \subseteq B(a, r)$ and hence $B(a, r)$ is open.

Next we claim that $\overline{B}(a, r)$ is closed, that is $\overline{B}(a, r)^c$ is open. Let $b \in \overline{B}(a, r)^c$, that is let $b \in \mathbb{R}^n$ with $b \notin \overline{B}(a, r)$. Since $b \notin \overline{B}(a, r)$ we have $|b - a| > r$. Let $s = |b - a| - r > 0$. Let $x \in B(b, s)$ and note that $|x - b| < s$. Then we have

$$|b - a| = |b - x + x - a| \leq |b - x| + |x - a| < s + |x - a|$$

and so $|x - a| > |b - a| - s = r$. Since $|x - a| > r$ we have $x \notin \overline{B}(a, r)$ and so $x \in \overline{B}(a, r)^c$. This shows that $B(b, s) \subseteq \overline{B}(a, r)^c$ and it follows that $\overline{B}(a, r)^c$ is open and hence that $\overline{B}(a, r)$ is closed.

2.13 Theorem: *(Basic Properties of Open Sets)*

(1) The sets $\emptyset$ *and* $\mathbb{R}^n$ *are open in* $\mathbb{R}^n$.
(2) If S *is a set of open sets then the union* $\bigcup S = \bigcup\limits_{U\in S} U$ *is open.*
(3) If S *is a finite set of open sets then the intersection* $\bigcap S = \bigcap\limits_{U\in S} U$ *is open.*

Proof: The empty set is open because any statement of the form "for all $x\in\emptyset$ F" (where F is any statement) is considered to be true (by convention). The set $\mathbb{R}^n$ is open because given $a \in \mathbb{R}^n$ we can choose any value of $r > 0$ and then we have $B(a,r) \subseteq \mathbb{R}^n$ by the definition of $B(a,r)$. This proves Part (1).

To prove Part (2), let S be any set of open sets. Let $a \in \bigcup S = \bigcup_{U\in S} U$. Choose an open set $U \in S$ such that $a \in U$. Since U is open we can choose $r > 0$ such that $B(a,r) \subseteq U$. Since $U \in S$ we have $U \subseteq \bigcup S$. Since $B(a,r) \subseteq U$ and $U \subseteq \bigcup S$ we have $B(a,r) \subseteq \bigcup S$. Thus $\bigcup S$ is open, as required.

To prove Part (3), let S be a finite set of open sets. If $S = \emptyset$ then we use the convention that $\bigcap S = \mathbb{R}^n$, which is open. Suppose that $S \neq \emptyset$, say $S = \{U_1, U_2, \cdots, U_m\}$ where each U_k is an open set. Let $a \in \bigcap S = \bigcap_{k=1}^m U_k$. For each index k, since $a \in U_k$ we can choose $r_k > 0$ so that $B(a, r_k) \subseteq U_k$. Let $r = \min\{r_1, r_2, \cdots, r_m\}$. Then for each index k we have $B(a,r) \subseteq B(a,r_k) \subseteq U_k$. Since $B(a,r) \subseteq U_k$ for every index k, it follows that $B(a,r) \subseteq \bigcap_{k=1}^m U_k = \bigcap S$. Thus $\bigcap S$ is open, as required.

2.14 Theorem: *(Basic Properties of Closed Sets)*

(1) The sets $\emptyset$ *and* $\mathbb{R}^n$ *are closed in* $\mathbb{R}^n$.
(2) If S *is a set of closed sets then the intersection* $\bigcap S = \bigcap\limits_{K\in S} K$ *is closed.*
(3) If S *is a finite set of closed sets then the union* $\bigcup S = \bigcup\limits_{K\in S} K$ *is closed.*

Proof: The proof is left as an exercise

2.15 Definition: Let $A \subseteq \mathbb{R}^n$. The **interior** and the **closure** of A (in $\mathbb{R}^n$) are the sets

$$A^0 = \bigcup \left\{ U \subseteq \mathbb{R}^n \middle| U \text{ is open, and } U \subseteq A \right\},$$
$$\overline{A} = \bigcap \left\{ K \subseteq \mathbb{R}^n \middle| K \text{ is closed and } A \subseteq K \right\}.$$

2.16 Theorem: *Let* $A \subseteq \mathbb{R}^n$.

(1) The interior of A *is the largest open set which is contained in* A. *In other words,* $A^0 \subseteq A$ *and* A^0 *is open, and for every open set* U *with* $U \subseteq A$ *we have* $U \subseteq A^0$.

(2) The closure of A *is the smallest closed set which contains* A. *In other words,* $A \subseteq \overline{A}$ *and* $\overline{A}$ *is closed, and for every closed set* K *with* $A \subseteq K$ *we have* $\overline{A} \subseteq K$.

Proof: Note that A^0 is open by Part (2) of Theorem 2.13, because A^0 is equal to the union of a set of open sets. Also note that $A^0 \subseteq A$ because A^0 is equal to the union of a set of subsets of A. Finally note that for any open set U with $U \subseteq A$ we have $U \in S$ so that $U \subseteq \bigcup S = A^0$. This completes the proof of Part (1), and the proof of Part (2) is similar.

2.17 Corollary: *Let* $A \subseteq \mathbb{R}^n$.
(1) $(A^0)^0 = A^0$ *and* $\overline{\overline{A}} = \overline{A}$.
(2) A *is open if and only if* $A = A^0$
(3) A *is closed if and only if* $A = \overline{A}$.

Proof: The proof is left as an exercise.

2.18 Definition: Let $A \subseteq \mathbb{R}^n$. An **interior point** of A is a point $a \in A$ such that for some $r > 0$ we have $B(a,r) \subseteq A$. A **limit point** of A is a point $a \in \mathbb{R}^n$ such that for every $r > 0$ we have $B^*(a,r) \cap A \neq \emptyset$. An **isolated point** of A is a point $a \in A$ which is not a limit point of A. A **boundary point** of A is a point $a \in \mathbb{R}^n$ such that for every $r > 0$ we have $B(a,r) \cap A \neq \emptyset$ and $B(a,r) \cap A^c \neq \emptyset$. The set of limit points of A is denoted by A'. The **boundary** of A, denoted by ∂A, is the set of all boundary points of A.

2.19 Theorem: *(Properties of Interior, Limit and Boundary Points) Let $A \subseteq \mathbb{R}^n$.*
(1) A^0 is equal to the set of all interior points of A.
(2) A is closed if and only if $A' \subseteq A$.
(3) $\overline{A} = A \cup A'$.
(4) $\partial A = \overline{A} \setminus A^0$.

Proof: We leave the proofs of Parts (1) and (4) as exercises. To prove Part (2) note that when $a \notin A$ we have $B(a,r) \cap A = B^*(a,r) \cap A$ and so

$$\begin{aligned} A \text{ is closed} &\iff A^c \text{ is open} \\ &\iff \forall a \in A^c \ \exists r > 0 \ B(a,r) \subseteq A^c \\ &\iff \forall a \in \mathbb{R}^n \ \big(a \notin A \implies \exists r > 0 \ B(a,r) \subseteq A^c \\ &\iff \forall a \in \mathbb{R}^n \ \big(a \notin A \implies \exists r > 0 \ B(a,r) \cap A = \emptyset\big) \\ &\iff \forall a \in \mathbb{R}^n \ \big(a \notin A \implies \exists r > 0 \ B^*(a,r) \cap A = \emptyset\big) \\ &\iff \forall a \in \mathbb{R}^n \ \big(\forall r > 0 \ B^*(a,r) \cap A \neq \emptyset \implies a \in A\big) \\ &\iff \forall a \in \mathbb{R}^n \ \big(a \in A' \implies a \in A\big) \\ &\iff A' \subseteq A. \end{aligned}$$

To prove Part (3) we shall prove that $A \cup A'$ is the smallest closed set which contains A. It is clear that $A \cup A'$ contains A. We claim that $A \cup A'$ is closed, that is $(A \cup A')^c$ is open. Let $a \in (A \cup A')^c$, that is let $a \in \mathbb{R}^n$ with $a \notin A$ and $a \notin A'$. Since $a \notin A'$ we can choose $r > 0$ so that $B(a,r) \cap A = \emptyset$. We claim that because $B(a,r) \cap A = \emptyset$ it follows that $B(a,r) \cap A' = \emptyset$. Suppose, for a contradiction, that $B(a,r) \cap A' \neq \emptyset$. Choose $b \in B(a,r) \cap A'$. Since $b \in B(a,r)$ and $B(a,r)$ is open, we can choose $s > 0$ so that $B(b,s) \subseteq B(a,r)$. Since $b \in A'$ it follows that $B(b,s) \cap A \neq \emptyset$. Choose $x \in B(b,s) \cap A$. Then we have $x \in B(b,s) \subseteq B(a,r)$ and $x \in A$ and so $x \in B(a,r) \cap A$, which contradicts the fact that $B(a,r) \cap A = \emptyset$. Thus $B(a,r) \cap A' = \emptyset$, as claimed. Since $B(a,r) \cap A = \emptyset$ and $B(a,r) \cap A' = \emptyset$ it follows that $B(a,r) \cap (A \cup A') = \emptyset$ hence $B(a,r) \subseteq (A \cup A')^c$. Thus proves that $(A \cup A')^c$ is open, and hence $A \cup A'$ is closed.

It remains to show that for every closed set K with $A \subseteq K$ we have $A \cup A' \subseteq K$. Let K be a closed set in $\mathbb{R}^n$ with $A \subseteq K$. Note that since $A \subseteq K$ it follows that $A' \subseteq K'$ because if $a \in A'$ then for all $r > 0$ we have $B(a,r) \cap A \neq \emptyset$ hence $B(a,r) \cap K \neq \emptyset$ and so $a \in K'$. Since K is closed we have $K' \subseteq K$ by Part (2). Since $A' \subseteq K'$ and $K' \subseteq K$ we have $A' \subseteq K$. Since $A \subseteq K$ and $A' \subseteq K$ we have $A \cup A' \subseteq K$, as required. This completes the proof of Part (3).

2.20 Definition: Let $A \subseteq \mathbb{R}^n$. For sets $U, V \subseteq \mathbb{R}^n$, we say that U and V **separate** A when

$$U \cap A \neq \emptyset \,,\ V \cap A \neq \emptyset \,,\ U \cap V = \emptyset \text{ and } A \subseteq U \cup V.$$

We say that A is **connected** when there do not exist open sets U and V in $\mathbb{R}^n$ which separate A. We say that A is **disconnected** when it is not connected, that is when there do exist open sets U and V in $\mathbb{R}^n$ which separate A.

2.21 Theorem: *The connected sets in* $\mathbb{R}$ *are the intervals, that is the sets of one of the forms*

$$(a,b)\,,\,[a,b)\,,\,(a,b]\,,\,[a,b]\,,\,(a,\infty)\,,\,[a,\infty)\,,\,(-\infty,b)\,,\,(-\infty,b]\,,\,(-\infty,\infty)$$

for some $a,b\in\mathbb{R}$ *with* $a\leq b$. *We include the case that* $a=b$ *in order to include the degenerate intervals* $\emptyset=(a,a)$ *and* $\{a\}=[a,a]$.

Proof: We use the fact that the intervals in $\mathbb{R}$ are the sets with the intermediate value property (a set $A\subseteq\mathbb{R}$ has **the intermediate value property** when for all $a,b,\in A$ and all $x\in\mathbb{R}$, if $a<x<b$ then $x\in A$). Let $A\subseteq\mathbb{R}$. Suppose that A is not an interval. Then A does not have the intermediate value property so we can choose $a,b\in A$ and $u\in\mathbb{R}$ with $a<u<b$. Then $U=(-\infty,u)$ and $V=(u,\infty)$ separate A and so A is disconnected.

Suppose, conversely, that A is disconnected. Choose open sets U and V which separate A. Choose $a\in U$ and $b\in V$. Note that $a\neq b$ since $U\cap V=\emptyset$. Suppose that $a<b$ (the case that $b<a$ is similar). Let $u=\sup\big(U\cap[a,b]\big)$. Note that $u\neq a$ since we can choose $\delta>0$ such that $[a,a+\delta)\subseteq U\cap[a,b]$ and then we have $u=\sup\big(U\cap[a,b]\big)\geq a+\delta$. Note that $u\neq b$ since we can choose $\delta>0$ such that $(b-\delta,b]\subseteq V\cap[a,b]$ and then we have $u=\sup\big(U\cap[a,b]\big)\leq b-\delta$ since $U\cap V=\emptyset$. Thus we have $a<u<b$. Note that $u\notin U$ since if we had $u\in U$ we could choose $\delta>0$ such that $(u-\delta,u+\delta)\subseteq U\cap[a,b]$ which contradicts the fact that $u=\sup\big(U\cap[a,b]\big)$. Note that $u\notin V$ since if we had $u\in V$ then we could choose $\delta>0$ such that $(u-\delta,u+\delta)\subseteq V\cap[a,b]$ which contradicts the fact that $u=\sup\big(U\cap[a,b]\big)$ because $U\cap V=\emptyset$. Since $u\notin U$ and $u\notin V$ and $A\subseteq U\cap V$ we have $u\notin A$, so A does not have the intermediate value property, and so A is not an interval.

2.22 Definition: Let $A\subseteq\mathbb{R}^n$. An **open cover** of A is a set S of open sets in $\mathbb{R}^n$ such that $A\subseteq\bigcup S$. A **subcover** of an open cover S of A is a subset $T\subseteq S$ such that $A\subseteq\bigcup T$. We say that A is **compact** when every open cover of A has a finite subcover.

2.23 Exercise: Show that the set $A=\left\{\frac{1}{n}\middle|n\in\mathbb{Z}^+\right\}$ is not compact, but that the set $B=A\cup\{0\}$ is compact.

2.24 Theorem: *(The Nested Interval Theorem) Let* $I_0,I_1,I_2,\cdots$ *be nonempty, closed bounded intervals in* $\mathbb{R}$. *Suppose that* $I_0\supseteq I_1\supset I_2\supset\cdots$. *Then* $\bigcap\limits_{k=0}^{\infty}I_k\neq\emptyset$.

Proof: For each $k\geq 1$, let $I_k=[a_k,b_k]$ with $a_k<b_k$. For each k, since $I_{k+1}\subseteq I_k$ we have $a_k\leq a_{k+1}<b_{k+1}\leq b_{k+1}$. Since $a_k\geq a_{k+1}$ for all k, the sequence (a_k) is increasing. Since $a_k<b_k\leq b_{k-1}\leq\cdots\leq b_1$ for all k, the sequence (a_k) is bounded above by b_1. Since (a_k) is increasing and bounded above, it converges. Let $a=\sup\{a_k\}=\lim\limits_{k\to\infty}a_k$. Similarly, (b_k) is decreasing and bounded below by a_1, and so it converges. Let $b=\inf\{b_k\}=\lim\limits_{k\to\infty}b_k$. Fix $m\geq 1$. For all $k\geq m$ we have $a_m<b_m\leq b_{m+1}\leq\cdots\leq b_k$. Since $a_k\leq b_k$ for all k, by the Comparison Theorem we have $a\leq b$, and so the interval $[a,b]$ is not empty. Since (a_k) is increasing with $a_k\to a$, it follows (we leave the proof as an exercise) that $a_k\leq a$ for all $k\geq 1$. Similarly, we have $b_k\geq b$ for all $k\geq 1$ and so $[a,b]\subseteq[a_k,b_k]=I_k$. Thus $[a,b]\subseteq\bigcap\limits_{k=1}^{\infty}I_k$, and so $\bigcap\limits_{k=1}^{\infty}I_k\neq\emptyset$.

2.25 Definition: A **closed rectangle** in $\mathbb{R}^n$ is a set of the form

$$\begin{aligned}R&=[a_1,b_1]\times[a_2,b_2]\times\cdots\times[a_n,b_n]\\&=\big\{(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n\big|a_j\leq x_j\leq b_j\text{ for all }j\big\}.\end{aligned}$$

2.26 Theorem: *(Nested Rectangles) Let $R_1, R_2, R_3, \cdots$ be closed rectangles in $\mathbb{R}^n$ with $R_1 \supseteq R_2 \supseteq R_3 \supseteq \cdots$. Then*

$$\bigcap_{k=1}^{\infty} R_k \neq \emptyset.$$

Proof: Let $R_k = [a_{k,1}, b_{k,1}] \times [a_{k,2}, b_{k,2}] \times \cdots \times [a_{k,n}, b_{k,n}]$. Since $R_1 \supseteq R_2 \supseteq \cdots$ it follows that for each index j with $1 \leq j \leq n$ we have $[a_{1,j}, b_{1,j}] \supseteq [a_{2,j}, b_{2,j}] \supseteq \cdots$. By the Nested Interval Theorem, for each index j we can choose $u_j \in \bigcap\limits_{k=1}^{\infty} [a_{k,j}, b_{k,j}]$. Then for $u = (u_1, u_2, \cdots, u_n)$ we have $u \in \bigcap\limits_{k=1}^{\infty} R_k$.

2.27 Theorem: *(Compactness of Rectangles) Every closed rectangle in $\mathbb{R}^n$ is compact.*

Proof: Let $R = I_1 \times I_2 \times \cdots \times I_n$ where $I_j = [a_j, b_j]$ with $a_j \leq b_j$. Let d be the diameter of R, that is $d = \text{diam}(R) = \big(\sum\limits_{j=1}^{n} (b_j - a_j)^2 \big)^{1/2}$. Let S be an open cover of R. Suppose, for a contradiction, that S does not have a finite subset which covers R. Let $a_{1,j} = a_j$, $b_{1,j} = b_j$, $I_{1,j} = I_j = [a_{1,j}, b_{1,j}]$ and $R_1 = R = I_{1,1} \times \cdots \times I_{1,n}$. Recursively, we construct rectangles $R = R_1 \supseteq R_2 \supseteq R_3 \supseteq \cdots$, with $R_k = I_{k,1} \times \cdots \times I_{k,n}$ where $I_{k,j} = [a_{k,j}, b_{k,j}]$, and $d_k = \text{diam}(R_k) = \big(\sum\limits_{j=1}^{n} (b_{k,j} - a_{k,j})^2 \big)^{1/2} = \frac{d}{2^{k-1}}$, such that the open cover S does not have a finite subset which covers any of the rectangles R_k. We do this recursive construction as follows. Having constructed one of the rectangles R_k, we partition each of the intervals $I_{k,j} = [a_{k,j}, b_{k,j}]$ into the two equal-sized subintervals $[a_{k,j}, \frac{a_{k,j}+b_{k,j}}{2}]$ and $[\frac{a_{k,j}+b_{k,j}}{2}, b_{k,j}]$, and we thereby partition the rectangle R_k into 2^n equal-sized sub-rectangles. We choose R_{k+1} to be equal to one of these 2^n sub-rectangles with the property that the open cover S does not have a finite subset which covers R_{k+1} (if each of the 2^n sub-rectangles could be covered by a finite subset of S then the union of theses 2^n finite subsets would be a finite subset of S which covers R_k).

By the Nested Rectangles Theorem, we can choose an element $u \in \bigcap\limits_{k=1}^{\infty} R_k$. Since $u \in R$ and S covers R we can choose an open set $U \in S$ such that $u \in U$. Since U is open we can choose $r > 0$ such that $B(u, r) \subseteq U$. Since $d_k \to 0$ we can choose k so that $d_k < r$. Since $u \in R_k$ and $\text{diam} R_k = d_k < r$ we have $R_k \subseteq B(u, r) \subseteq U$. Thus S does have a finite subset, namely $\{U\}$, which covers R_k, giving the desired contradiction.

2.28 Theorem: *Let $A \subseteq K \subseteq \mathbb{R}^n$. If A is closed and K is compact then A is compact.*

Proof: Suppose that A is closed in $\mathbb{R}^n$ and that K is compact. Let S be an open cover of A. Let $A^c = \mathbb{R}^n \setminus A$. Since $A \subseteq \bigcup S$ we have $\bigcup S \cup \{A^c\} = \mathbb{R}^n$ and so $S \cup \{A^c\}$ is an open cover of K. Since K is compact, we can choose a finite subset $T \subseteq S \cup \{A^c\}$ with $K \subseteq \bigcup T$. Since $A \subseteq K \subseteq \bigcup T$ we also have $A \subseteq \bigcup \big(T \setminus \{A^c\}\big)$. Thus the open cover S of A does have a finite subcover, namely $T \setminus \{A^c\}$, and so A is compact, as required.

2.29 Theorem: *(The Heine-Borel Theorem) Let $A\subseteq\mathbb{R}^n$. Then A is compact if and only if A is closed and bounded.*

Proof: Suppose that A is compact. Suppose, for a contradiction, that A is not bounded. For each $k\in\mathbb{Z}^+$ let $U_k=B(0,k)$ and let $S=\big\{U_k\big|k\in\mathbb{Z}^+\big\}$. Then $\bigcup S=\mathbb{R}^n$ so S is an open cover of A. Let T be any finite subset of S. If $T=\emptyset$ then $\bigcup T=\emptyset$ and $A\not\subseteq\bigcup T$. Suppose that $T\neq\emptyset$, say $T=\big\{U_{k_1},U_{k_2},\cdots,U_{k_m}\big\}$ with $k_1<k_2<\cdots<k_m$. Since $U_{k_1}\subseteq U_{k_2}\subseteq\cdots\subseteq U_{k_m}$ we have $\bigcup T=\bigcup_{i=1}^m U_{k_i}=U_{k_m}=B(0,k_m)$. Since A is not bounded we have $A\not\subseteq B(0,k_m)$ and so $A\not\subseteq\bigcup T$. This shows that the open cover S has no finite subcover T, which contradicts the fact that A is compact.

Next suppose, for a contradiction, that A is not closed. By Part (2) of Theorem 2.19, it follows that $A'\not\subseteq A$. Choose $a\in A'$ with $a\notin A$. For each $k\in\mathbb{Z}^+$ let U_k be the open set $U_k=\overline{B}\big(a,\frac{1}{k}\big)^c=\big\{x\in\mathbb{R}^n\big||x-a|>\frac{1}{k}\big\}$ and let $S=\big\{U_k\big|k\in\mathbb{Z}^+\big\}$. Note that $\bigcup S=\mathbb{R}^n\setminus\{a\}$ so S is an open cover of A. Let T be any finite subset of S. If $T=\emptyset$ then $\bigcup T=\emptyset$ so $A\not\subseteq\bigcup T$ (since A is not closed so $A\neq\emptyset$). Suppose that $T\neq\emptyset$, say $T=\big\{U_{k_1},U_{k_2},\cdots,U_{k_m}\big\}$ with $k_1<k_2<\cdots<k_m$. Since $U_{k_1}\subseteq U_{k_2}\subseteq\cdots\subseteq U_{k_m}$ we have $\bigcup T=\bigcup_{i=1}^m U_{k_i}=U_{k_m}=\overline{B}\big(a,\frac{1}{k_m}\big)^c$. Since a is a limit point of A we have $B\big(a,\frac{1}{k_m}\big)\neq\emptyset$ hence $\overline{B}\big(a,\frac{1}{k_m}\big)\cap A\neq\emptyset$ and so $A\not\subseteq\overline{B}\big(a,\frac{1}{k_m}\big)^c$, hence $A\not\subseteq\bigcup T$. This shows that the open cover S has no finite subcover T, which again contradicts the fact that A is compact.

Suppose, conversely, that A is closed and bounded. Since A is bounded we can choose $r>0$ so that $A\subseteq B(0,r)$. Let R be the closed rectangle $R=\big\{x\in\mathbb{R}^n\big||x_k|\leq r\text{ for all }k\big\}$. Note that $B(0,r)\subseteq R$ since when $x=(x_1,\cdots,x_n)\in B(0,r)$, for each index k we have

$$|x_k|=\big({x_k}^2\big)^{1/2}\leq\Big(\sum_{i=1}^n{x_i}^2\Big)^{1/2}=|x|<r.$$

Since A is closed and $A\subseteq R$ and R is compact, it follows that A is compact, by the above theorem.

2.30 Definition: Let $P\subseteq\mathbb{R}^n$. For $a\in P$ and $0<r\in\mathbb{R}$ we define the **open ball in** P and the **closed ball in** P centred at a of radius r to be the sets

$$B_P(a,r)=\big\{x\in P\,\big|\,|x-a|<r\big\}=B(a,r)\cap P,$$
$$\overline{B}_P(a,r)=\big\{x\in P\,\big|\,|x-a|\leq r\big\}=\overline{B}(a,r)\cap P.$$

For $A\subseteq P\subseteq\mathbb{R}^n$, we say A is **open in** P when for every $a\in A$ there exists $r>0$ such that $B_P(a,r)\subseteq A$, and we say A is **closed in** P when $A^c=P\setminus A$ is open in P.

2.31 Theorem: *Let $A\subseteq P\subseteq\mathbb{R}^n$.*

(1) A is open in P if and only if there exists an open set U in $\mathbb{R}^n$ such that $A=U\cap P$.
(2) A is closed in P if and only if there exists a closed set K in $\mathbb{R}^n$ such that $A=K\cap P$.

Proof: To prove Part (1), suppose first that A is open in P. For each $a\in A$, choose $r_a>0$ so that $B(a,r_a)\cap P\subseteq A$, and let $U=\bigcup_{a\in A}B(a,r_a)$. Since U is equal to the union of a set of open sets in $\mathbb{R}^n$, it follows that U is open in $\mathbb{R}^n$. Note that $A\subseteq U\cap P$ and, since $B(a,r_a)\cap P\subseteq A$ for every $a\in A$, we also have $U\cap P=\Big(\bigcup_{a\in U}B(a,r_a)\Big)\cap P=\bigcup_{a\in A}\big(B(a,r_a)\cap P\big)\subseteq A$. Thus $A=U\cap P$, as required.

Suppose, conversely, that $A=U\cap P$ with U open in $\mathbb{R}^n$. Let $a\in A$. Since $a\in A=U\cap P$, we also have $a\in U$. Since $a\in U$ and U is open in $\mathbb{R}^n$ we can choose $r>0$ so that $B(a,r)\subseteq U$. Since $B(a,r)\subseteq U$ and $U\cap P=A$ we have $B(a,r)\cap P\subseteq U\cap P=A$, as required.

To prove Part (2), suppose first that A is closed in P. Let B be the complement of A in P, that is $B = P \setminus A$. Then B is open in P. Choose an open set U in $\mathbb{R}^n$ such that $B = U \cap P$. Let K be the complement of U in $\mathbb{R}^n$, that is $K = \mathbb{R}^n \setminus U$. Then $A = K \cap P$ since for $x \in \mathbb{R}^n$ we have $x \in A \iff \big(x \in P \text{ and } x \notin B\big) \iff \big(x \in P \text{ and } x \notin U \cap P\big) \iff \big(x \in P \text{ and } x \notin U\big) \iff \big(x \in P \text{ and } x \in K\big) \iff x \in K \cap P$.

Suppose, conversely, that K is a closed set in P with $A = K \cap P$. Let B be the complement of A in P, that is $B = P \setminus A$, and let U be the complement of K in P, that is $U = P \setminus K$, and note that U is open in P. Then we have $B = U \cap P$ since for $x \in P$ we have $x \in B \iff \big(x \in P \text{ and } x \notin A\big) \iff \big(x \in P \text{ and } x \notin K \cap P\big) \iff \big(x \in P \text{ and } x \notin K\big) \iff \big(x \in P \text{ and } x \in U\big) \iff x \in U \cap P$. Since U is open in P and $B = U \cap P$ we know that B is open in P. Since B is open in P, its complement $A = P \setminus B$ is closed in P.

2.32 Theorem: *Let $A \subseteq P \subseteq \mathbb{R}^n$. Define A to be* **connected in** *P when there do not exists sets $E, F \subseteq P$ which are open in P and which separate A. Define A to be* **compact in** *P when for every set S of open sets in P such that $A \subseteq \bigcup S$ there exists a finite subset $T \subseteq S$ such that $A \subseteq \bigcup T$. Then*

(1) A is connected in P if and only if A is connected in $\mathbb{R}^n$, and
(2) A is compact in P if and only if A is compact in $\mathbb{R}^n$.

Proof: We prove. Part (1) and leave the proof of Part (2) as an exercise. Suppose that A is not connected in $\mathbb{R}^n$. Choose open sets U and V in $\mathbb{R}^n$ which separate A, that is $U \cap A \neq \emptyset$, $V \cap A \neq \emptyset$, $U \cap V = \emptyset$ and $A \subseteq U \cup V$. Let $E = U \cap P$ and $F = V \cap P$. Note that E and F are open in P and E and F separate A.

Suppose, conversely, that there exist sets $E, F \subseteq P$ which are open in P and which separate A, that is $A \cap E \neq \emptyset$, $A \cap F \neq \emptyset$, $E \cap F = \emptyset$ and $A \subseteq E \cup F$. Choose open sets $U, V \subseteq \mathbb{R}^n$ such that $E = U \cap P$ and $F = V \cap P$. Note that it is possible that $U \cap V \neq \emptyset$ and so U and V might not separate A in $\mathbb{R}^n$. For this reason, we shall construct open subsets $U_0 \subseteq U$ and $V_0 \subseteq V$ which do separate A in $\mathbb{R}^n$. For each $a \in E$ choose $r_a > 0$ such that $B(a, 2r_a) \subseteq U$ and then let $U_0 = \bigcup_{a \in E} B(a, r_a)$. Note that U_0 is open in $\mathbb{R}^n$ (since it is a union of open sets in $\mathbb{R}^n$) and that we have $E \subseteq U_0 \subseteq U$. Similarly, for each $b \in F$ choose $s_b > 0$ so that $B(b, 2s_b) \subseteq V$, and then let $V_0 = \bigcup_{b \in F} B(b, s_b)$. Note that V_0 is open in $\mathbb{R}^n$ and $F \subseteq V_0 \subseteq V$. We claim that the open sets U_0 and V_0 separate A in $\mathbb{R}^n$. Since $E \subseteq U_0$ and $F \subseteq V_0$ we have $\emptyset \neq A \cap E \subseteq A \cap U_0$, $\emptyset \neq A \cap F \subseteq A \cap V_0$ and $A \subseteq E \cup F \subseteq U_0 \cup V_0$. It remains to show that $U_0 \cap V_0 = \emptyset$. Suppose, for a contradiction, that $U_0 \cap V_0 \neq \emptyset$. Choose $x \in U_0 \cap V_0$. Since $x \in U_0 = \bigcup_{a \in E} B(a, r_a)$ we can choose $a \in E$ such that $x \in B(a, r_a)$. Similarly, we can choose $b \in F$ so that $x \in B(b, s_b)$. Suppose that $r_a \geq s_b$ (the case that $s_b \geq r_a$ is similar). By the Triangle Inequality, it follows that $|b - a| \leq |b - x| + |x - a| < s_b + r_a \leq 2r_a$ and so we have $b \in B(a, 2r_a) \subseteq U$. Since $b \in F \subseteq P$ and $b \in U$ we have $b \in U \cap P = E$. Thus we have $b \in E \cap F$ which contradicts the fact that $E \cap F = \emptyset$, and so $U_0 \cap V_0 = \emptyset$, as required.

2.33 Corollary: *A set $A \subseteq \mathbb{R}^n$ is connected (in $\mathbb{R}^n$) if and only if the only subsets of A which are both open and closed in A are the sets $\emptyset$ and A.*

Proof: We leave it as an exercise to show that this follows from the above theorem by taking $A = P$.