1. Appendix. Bilinear Forms

1.1 Definition: Let U, V and W be vector spaces over a field $\mathbb{F}$ and let $L : U \times V \to W$. We say that L is **bilinear** when

$$\begin{aligned} L(x_1 + x_2, y) &= L(x_1, y) + L(x_2, y) \quad , \quad L(tx, y) = t\, L(x, y)\,, \\ L(x, y_1 + y_2) &= L(x, y_1) + L(x, y_2) \quad \text{and} \quad L(x, ty) = t\, L(x, y) \end{aligned}$$

for all $x, x_1, x_2 \in U$, and all $y, y_1, y_2 \in V$ and all $t \in \mathbb{F}$. For a bilinear map $L : U \times U \to W$, we say that L is **symmetric** when $L(y, x) = L(x, y)$ for all $x, y \in U$, and we say that L is **alternating** (or **skew-symmetric**) when $L(y, x) = -L(x, y)$ for all $x, y \in U$. A bilinear map $L : U \times U \to \mathbb{F}$ is called a **bilinear form** on U.

1.2 Example: For any field $\mathbb{F}$, the **dot product** $\boldsymbol{\cdot} : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}$ given by $u \cdot v = v^T u$ is a symmetric bilinear form on $\mathbb{F}^n$, and the **cross product** $\times : \mathbb{F}^3 \times \mathbb{F}^3 \to \mathbb{F}$ given by $u \times v = \big(u_2 v_3 - u_3 v_2\,,\, u_3 v_1 - u_1 v_3\,,\, u_1 v_2 - u_2 v_1\big)$ is an alternating bilinear form on $\mathbb{F}^3$.

1.3 Note: Let U, V and W be vector spaces over a field $\mathbb{F}$. Given bases $\mathcal{A}$ and $\mathcal{B}$ for U and V, A bilinear map $L : U \times V \to W$ is uniquely determined by the values $L(u, v) \in W$ with $u \in \mathcal{A}$ and $v \in \mathcal{B}$. Indeed, given $x \in U$ and $y \in V$, say $x = \sum_{i=1}^{n} s_i u_i$ and $y = \sum_{j=1}^{m} t_j v_j$ with $u_i \in \mathcal{A}$, $v_j \in \mathcal{B}$ and $s_i, t_j \in \mathbb{F}$, we have

$$L(x, y) = L\Big(\sum_{i=1}^{n} s_i u_u\,,\, \sum_{j=1}^{m} t_j v_j\Big) = \sum_{1 \le i \le n, 1 \le j \le m} s_i t_j L(u_i, v_j).$$

1.4 Theorem: *(The Matrix of a Bilinear Map) Let U and V be finite dimensional vector spaces over a field $\mathbb{F}$. Let $\mathcal{A} = \{u_1, \cdots, u_k\}$ and $\mathcal{B} = \{v_1, \cdots, v_l\}$ be bases for U and V. Let $L : U \times V \to \mathbb{F}$ be a bilinear map. There exists a unique matrix $[L]^{\mathcal{A}}_{\mathcal{B}} \in M_{l \times k}(\mathbb{F})$ with the property that*

$$[y]_{\mathcal{B}}{}^T [L]^{\mathcal{A}}_{\mathcal{B}}\, [x]_{\mathcal{A}} = L(x, y)$$

for all $x \in U$ and $y \in V$.

Proof: First we prove uniqueness. Suppose that such a matrix $[L]^{\mathcal{A}}_{\mathcal{B}}$ exists. Let $A = [L]^{\mathcal{A}}_{\mathcal{B}}$. Then the entries of A are given by

$$A_{i,j} = e_i^T A\, e_j = [v_i]_{\mathcal{B}}{}^T [L]^{\mathcal{A}}_{\mathcal{B}}\, [u_j]_{\mathcal{A}} = L(u_j, v_i).$$

This shows that the matrix is unique.

To prove existence, given a bilinear map $L : U \times V \to \mathbb{F}$, we let $A \in M_{l \times k}(\mathbb{F})$ be the matrix with entries $A_{i,j} = L(u_j, v_i)$. Define $M : U \times V \to \mathbb{F}$ by $M(x, y) = [y]_{\mathcal{B}}{}^T A\, [x]_{\mathcal{A}}$. Note that M is bilinear and for all indices i and j we have

$$M(u_j, v_i) = [v_i]_{\mathcal{B}}{}^T A\, [u_j]_{\mathcal{A}} = e_i^T A\, e_j = A_{i,j} = L(u_j, v_i).$$

It follows from the above note that $M = L$, so we can take $[L]^{\mathcal{A}}_{\mathcal{B}} = A$.

1.5 Definition: The matrix $[L]^{\mathcal{A}}_{\mathcal{B}}$ in the above theorem is called the **matrix** of the bilinear map L with respect to the bases $\mathcal{A}$ and $\mathcal{B}$. For a bilinear form $L : U \times U \to \mathbb{F}$, we write $[L]_{\mathcal{A}} = [L]^{\mathcal{A}}_{\mathcal{A}}$.

1.6 Theorem: *Let U be a finite dimensional vector space over a field $\mathbb{F}$. Let L be a bilinear form on U. Then L is symmetric if and only $[L]_{\mathcal{A}}$ is symmetric for some, hence any, basis $\mathcal{A}$ for U.*

Proof: Suppose that L is symmetric. Let $\mathcal{A} = \{u_1, \cdots, u_n\}$ be any basis for U and let $A = [L]_{\mathcal{A}}$. Then for all indices i, j we have $A_{i,j} = L(u_j, u_i) = L(u_i, u_j) = A_{j,i}$, and so A is symmetric. Conversely, let $\mathcal{A} = \{u_1, \cdots, u_n\}$ be any basis for U, let $A = [L]_{\mathcal{A}}$, and suppose that A is symmetric. Let $x, y \in U$, say $x = \sum_{i=1}^{n} s_i u_i$ and $y = \sum_{i=1}^{n} t_j u_j$ with $s_i, t_j \in \mathbb{F}$. Then

$$\begin{aligned} L(x,y) &= L\Big(\sum_{i=1}^{n} s_i u_i\,,\ \sum_{i=1}^{n} t_j u_j\Big) = \sum_{1\le i,j\le n} s_i t_j\, L(u_i,u_j) = \sum_{1\le i,j\le n} s_i t_j A_{j,i} \\ &= \sum_{1\le i,j\le n} s_i t_j A_{i,j} = \sum_{1\le i,j\le n} s_i t_j L(u_j,u_i) = L\Big(\sum_{i=1}^{n} t_j u_j\,,\ \sum_{i=1}^{n} s_i u_i\Big) = L(y,x). \end{aligned}$$

1.7 Theorem: *(Change of Basis) Let U and V be finite dimensional vector spaces over a field $\mathbb{F}$. Let $L : U \times V \to \mathbb{F}$ be a bilinear map. Let $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{B}_1$, $\mathcal{B}_2$ be two bases for each of the vector spaces U and V. Then*

$$[L]_{\mathcal{B}_2}^{\mathcal{A}_2} = {[I]_{\mathcal{B}_1}^{\mathcal{B}_2}}^T\, [L]_{\mathcal{B}_1}^{\mathcal{A}_1}\, [I]_{\mathcal{A}_1}^{\mathcal{A}_2}.$$

Proof: For all $x \in U$ and $y \in V$ we have

$$\begin{aligned} {[y]_{\mathcal{B}_2}}^T [L]_{\mathcal{B}_2}^{\mathcal{A}_2}\, [x]_{\mathcal{A}_2} = \mathbb{F}(x,y) = {[y]_{\mathcal{B}_1}}^T [L]_{\mathcal{B}_1}^{\mathcal{A}_1}\, [x]_{\mathcal{A}_1} &= \Big([I]_{\mathcal{B}!}^{\mathcal{B}_2}[y]_{\mathcal{B}_2}\Big)^T [L]_{\mathcal{B}_1}^{\mathcal{A}_1}\Big([I]_{\mathcal{A}_1}^{\mathcal{A}_2}[x]_{\mathcal{A}_2}\Big) \\ &= {[y]_{\mathcal{B}_2}}^T\Big({[I]_{\mathcal{A}_1}^{\mathcal{A}_2}}^T [L]_{\mathcal{B}_1}^{\mathcal{A}_1} [I]_{\mathcal{A}_1}^{\mathcal{A}_2}\Big)[x]_{\mathcal{A}_2} \end{aligned}$$

and so, by the uniqueness of the matrix $[L]_{\mathcal{B}_2}^{\mathcal{A}_2}$, we have $[L]_{\mathcal{B}_2}^{\mathcal{A}_2} = {[I]_{\mathcal{B}_1}^{\mathcal{B}_2}}^T\, [L]_{\mathcal{B}_1}^{\mathcal{A}_1}\, [I]_{\mathcal{A}_1}^{\mathcal{A}_2}$.

1.8 Definition: Let U and V be finite dimensional vector spaces over a field $\mathbb{F}$, and let $L : U \times V \to \mathbb{F}$ be a bilinear map. We define the **rank** of L to be the rank of the matrix $[L]_{\mathcal{B}}^{\mathcal{A}}$ where $\mathcal{A}$ and $\mathcal{B}$ are any bases for U and V. Note that, by the above theorem, this definition does not depend on the choice of $\mathcal{A}$ and $\mathcal{B}$ (because multiplying a matrix, on the right or on the left, by an invertible matrix does not alter its rank).

1.9 Note: As a particular case of the above theorem, if U is a finite dimensional vector space over a field $\mathbb{F}$, L is a bilinear form on U, and $\mathcal{A}$ and $\mathcal{B}$ are two bases for U, and if we write $A = [L]_{\mathcal{A}}$, $B = [L]_{\mathcal{B}}$ and $P = [I]_{\mathcal{A}}^{\mathcal{B}}$, then we have $B = P^T A P$.

1.10 Definition: For $A, B \in M_n(\mathbb{F})$, we say that A and B are **congruent**, and we write $A \cong B$, when there exists an invertible matrix $P \in M_n(\mathbb{F})$ such that $B = P^T A P$.

1.11 Note: It is perhaps worth mentioning that congruent matrices do *not*, in general, share the same trace, determinant, or eigenvalues, and we do *not* define the trace, determinant, or eigenvalues of a bilinear map.

1.12 Theorem: *(Diagonalization of Symmetric Bilinear Forms) Let U be a finite dimensional vector space over a field $\mathbb{F}$ with $\mathrm{char}\,(\mathbb{F}) \neq 2$. Let $L : U \times U \to \mathbb{F}$ be a bilinear form on U. Then there exists a basis $\mathcal{A}$ for U such that $[L]_{\mathcal{A}}$ is diagonal if and only if L is symmetric.*

Proof: If $\mathcal{A}$ is a basis for U such that $[L]_{\mathcal{A}}$ is diagonal, then L is symmetric since its matrix $[L]_{\mathcal{A}}$ is symmetric. Conversely, suppose that L is symmetric. Choose a basis $\mathcal{A}_0$ for U, and let $A = [L]_{\mathcal{A}_0} \in M_n(\mathbb{F})$. Note that A is symmetric. We must show that A is congruent to a diagonal matrix. We describe an algorithm which uses elementary row and column operations to put the matrix A into diagonal form. Consider the element $A_{1,1}$. If $A_{1,1} \neq 0$ then we use the $(1,1)$ entry to eliminate the other entries on the first row and column by applying the row and column operations

$$R_k \mapsto R_k - \frac{A_{k,1}}{A_{1,1}}\, R_1 \ \text{ and } C_k \mapsto C_k - \frac{A_{1,k}}{A_{1,1}}\, C_1 .$$

Note that since A is symmetric, we have $A_{k,1} = A_{1,k}$, and it follows that, for each $k \geq 2$, the elementary matrices associated to the above row and column operations are the transposes of one another. If $A_{1,1} = 0$ and $A_{1,j} \neq 0$ for some $j \geq 2$, say $A_{1,k} \neq 0$, then first we use row and column operations to replace the $(1,1)$ entry by a non-zero element in $\mathbb{F}$ as follows: if $A_{k,k} \neq 0$ we use the operations

$$R_1 \leftrightarrow R_k \ \text{ and } \ C_1 \leftrightarrow C_k$$

to replace the $(1,1)$ entry by $A_{k,k}$, and if $A_{k,k} = 0$ then we use the operations

$$R_1 \mapsto R_1 + R_k \ \text{ and } \ C_1 \mapsto C_1 + C_k$$

to replace the $(1,1)$ entry by $A_{1,k} + A_{k,1} = 2A_{1,k}$ (which is non-zero since $\mathrm{char}\,(\mathbb{F}) \neq 2$). Then we use this new non-zero $(1,1)$ entry to eliminate the other entries on the first row and column, as above. Again, note that the elementary matrices associated to the above row and column operation are the transposes of one another. At this stage we have converted A to the congruent matrix ${P_1}^T A P_1 = \begin{pmatrix} d_1 & 0 \\ 0 & B \end{pmatrix}$ with $B \in M_{n-1}(\mathbb{F})$, where P_1 is the product of all the elementary column operation matrices. Since A is symmetric, the matrix ${P_1}^T A P$ is symmetric, and so the matrix $B \in M_{n-1}(\mathbb{F})$ is also symmetric. We now repeat the above procedure on the matrix B.

1.13 Corollary: *Let U be a finite dimensional vector space over a field $\mathbb{F}$ and let L be a symmetric bilinear form on U of rank r. Then there is a basis $\mathcal{A}$ for U such that such that $[L]_U = \mathrm{diag}(d_1, d_2, \cdots, d_n)$ for some $d_i \in \mathbb{F}$ with $d_i \neq 0$ for $1 \leq r$ and $d_i = 0$ for $i > r$.*

Proof: Choose a basis $\mathcal{A}_0$ so that $[L]_{\mathcal{A}_0}$ is diagonal, then, if necessary, perform the row and column operations $R_i \leftrightarrow R_j$ and $C_i \leftrightarrow C_j$ to rearrange the diagonal entries of the matrix.

1.14 Corollary: *Let U be a finite dimensional vector space over $\mathbb{C}$ and let L be a symmetric bilinear form on U of rank r. Then there exists a basis $\mathcal{A}$ for U such that*

$$[L]_{\mathcal{A}} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} .$$

Proof: Choose a basis $\mathcal{A}_0$ for U so that $[L]_{\mathcal{A}_0} = D = \mathrm{diag}(d_1, d_2, \cdots, d_n)$ with $d_i \neq 0$ for $1 \leq i \leq r$ and $d_i = 0$ for $r < i \leq n$. For $1 \leq i \leq r$, choose $c_i \in \mathbb{C}$ so that ${c_i}^2 = \frac{1}{d_i}$ and for $r < i \leq n$, choose $c_i = 1$, and then let $C = \mathrm{diag}(c_1, c_2, \cdots, c_n)$. Then D is congruent to the matrix $C^T D\, C$, which is in the required form.

1.15 Theorem: *(Sylvester's Law of Inertia) Let U be a finite dimensional vector space over $\mathbb{R}$ and let L be a symmetric bilinear form on U of rank r. Then there exists a basis $\mathcal{A}$ for U such that $[L]_\mathcal{A}$ is of the form*

$$[L]_\mathcal{A} = \begin{pmatrix} I_k & & \\ & -I_{r-k} & \\ & & 0 \end{pmatrix}$$

for some uniquely determined number k with $0 \leq k \leq r$.

Proof: We can choose a basis $\mathcal{A}_0$ for U so that $D = [L]_\mathcal{A}$ is diagonal, and we can order the diagonal entries so that $D = \text{diag}(d_1, d_2, \cdots, d_n$ with $d_i > 0$ for $1 \leq i \leq k$, $d_i < 0$ for $k < i \leq r$ and $d_i = 0$ for $r < i \leq n$. For $1 \leq i \leq k$ we choose $c_i = \frac{1}{\sqrt{d_i}}$, for $k < i \leq r$ we choose $c_i = \frac{1}{\sqrt{-d_i}}$ and for $k < i \leq n$ we choose $c_i = 1$, and then let $C = \text{diag}(c_1, c_2, \cdots, c_n)$. Then the matrix D is congruent to the matrix $C^T D C$ which is in the desired form.

It remains to show that the number of positive entries k is uniquely determined by L. Suppose, for a contradiction, that we can find two bases $\mathcal{A}$ and $\mathcal{B}$ for U such that

$$[L]_\mathcal{A} = \text{diag}\big(I_k, -I_{r-k}, 0\big) \;\text{ and }\; [L]_\mathcal{B} = \text{diag}\big(I_l, -I_{r-l}, 0\big)$$

with $k \neq l$, say $k < l$. Note that for $x \in U$, with say $x = \sum_{i=1}^n s_i u_i$, we have

$$L(x, u_j) = L\Big(\sum_{i=1}^n s_i u_i\,,\, u_j\Big) = \sum_{i=1}^n s_i L(u_i, u_j) = s_j L(u_j, u_j) = \begin{cases} s_j \text{ if } 1 \leq j \leq k \\ -s_j \text{ if } k < j \leq r, \\ 0 \text{ if } r < j \leq n \end{cases} \text{and}$$

and hence

$$L(x, x) = L\Big(x\,,\, \sum_{j=1}^n s_j u_j\Big) = \sum_{j=1}^n s_j L(x, u_j) = \sum_{j=1}^k {s_j}^2 - \sum_{j=k+1}^r {s_j}^2.$$

Similar formulas hold for $x \in U$ with $x = \sum_{i=1}^n t_i v_i$.

Consider the linear map $\phi : U \to \mathbb{R}^{k+r-l}$ given by

$$\phi(x) = \Big(L(x, u_1), L(x, u_2), \cdots, L(x, u_k), L(x, v_{l+1}), L(x, v_{l+2}), \cdots, L(x, v_r)\Big)^T.$$

Note that $\text{nullity}(\phi) = n - \text{rank}(\phi) \geq n - (k + r - l) = (n - r) + (l - k) > n - r$. Since $\text{nullity}(\phi) > n - r = \dim \text{Span}\{u_{r+1}, u_{r+2}, \cdots, u_n\}$, we can choose an element $x \in \text{Null}(\phi)$ with $x \notin \text{Span}\{u_{k+1}, \cdots, u_n\}$. Choose such an element x and write $x = \sum_{i=1}^n s_i u_i = \sum_{i=1}^n t_i v_i$. Since $x \in \text{Null}(\phi)$ we have $s_i = L(x, u_i) = 0$ for $1 \leq i \leq k$ and $t_i = -L(x, v_i) = 0$ for $l < i \leq r$. Since $x \notin \text{Span}\{u_{r+1}, \cdots, u_n\}$, we must have $s_i \neq 0$ for some $1 \leq i \leq r$. Thus we have $s_i = 0$ for all $1 \leq i \leq k$ and $s_i \neq 0$ for some $1 \leq i \leq r$, which implies that $L(x, x) = \sum_{i=1}^k {s_i}^2 - \sum_{i=k+1}^r {s_i}^2 < 0$, but we also have $t_i = 0$ for all $l < i \leq r$ which implies that $L(x, x) = \sum_{i=1}^l {t_i}^2 - \sum_{i=l+1}^r {t_i}^2 \geq 0$, giving the desired contradiction.

1.16 Definition: For a bilinear form $L : U \times U \to \mathbb{R}$, the number k in the above theorem is called the **index** of L, and the pair $(k, r-k)$ is called the **signature** of L.

1.17 Note: Let U be a finite dimensional inner product space over $\mathbb{R}$ and let L be a symmetric bilinear form on U. Let $\mathcal{A}$ be any basis for U and let $A = [L]_{\mathcal{A}}$. Since A is symmetric, it is orthogonally diagonalizable, so there exists an orthogonal matrix P such that $P^T AP = D = \text{diag}(\lambda_1, \cdots, \lambda_n)$ and the diagonal entries λ_i are the eigenvalues of A (repeated according to multiplicity). By Sylvester's Theorem the number of indices i for which $\lambda_i > 0$ is equal to the index of L, and does not depend on the choice of basis $\mathcal{A}$.

1.18 Definition: Let U be a vector space over $\mathbb{R}$ and let $L : U \times U \to \mathbb{R}$ be a symmetric bilinear form. Then

(1) L is **positive definite** when $L(x,x) > 0$ for all $0 \neq x \in U$,
(2) L is **positive semidefinite** when $L(x,x) \geq 0$ for all $x \in U$,
(3) L is **negative definite** when $L(x,x) < 0$ for all $0 \neq x \in U$,
(4) L is **negative semidefinite** when $L(x,x) \leq 0$ for all $x \in U$, and
(5) L is **indefinite** when there exist $x, y \in U$ with $L(x,x) > 0$ and $L(y,y) < 0$.

1.19 Note: Let Ube an n-dimensional vector space over $\mathbb{R}$ and let $L : U \times U \to \mathbb{R}$ be a symmetric bilinear form. Let $\mathcal{A}$ be a basis for U and let $A = [L]_{\mathcal{A}}$. Then

$$\begin{aligned} L \text{ is positive definite } &\iff L(u,u) > 0 \text{ for all } 0 \neq u \in U \\ &\iff [u]_{\mathcal{A}}{}^T [L]_{\mathcal{A}} [u]_{\mathcal{A}} > 0 \text{ for all } 0 \neq u \in U \\ &\iff x^T Ax > 0 \text{ for all } 0 \neq x \in \mathbb{R}^n. \end{aligned}$$

Similarly, L is positive semidefinite if and only if $x^T Ax \geq 0$ for all $x \in \mathbb{R}^n$, and so on.

1.20 Definition: For a symmetric matrix $A \in M_n(\mathbb{R})$,

(1) A is **positive definite** when $x^T Ax > 0$ for all $0 \neq x \in \mathbb{R}^n$,
(2) A is **positive semidefinite** when $x^T Ax \geq 0$ for all $x \in \mathbb{R}^n$,
(3) A is **negative definite** when $x^T Ax < 0$ for all $0 \neq x \in \mathbb{R}^n$,
(4) A is **negative semidefinite** when $x^T Ax \leq 0$ for all $x \in \mathbb{R}^n$, and
(5) A is **indefinite** when there exist $x, y \in \mathbb{R}^n$ with $x^T Ax > 0$ and $x^T Ax < 0$.

1.21 Theorem: *(The Characterization of Definiteness by Eigenvalues) Let U be an n-dimensional vector space over $\mathbb{R}$ and let $L : U \times U \to \mathbb{R}$ be a symmetric bilinear form. Let $\mathcal{A}$ be a basis for U and let $A = [L]_{\mathcal{A}} \in M_n(\mathbb{R})$. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of A. Then*

(1) L is positive definite $\iff \lambda_i > 0$ for all $i \iff$ $\text{index}\,(L) = \text{rank}(L) = \dim(U)$,
(2) L is positive semidefinite $\iff \lambda_i \geq 0$ for all $i \iff$ $\text{index}\,(L) = \text{rank}(L)$,
(3) L is negative definite $\iff \lambda_i < 0$ for all $i \iff$ $\text{index}\,(L) = 0$ *and* $\text{rank}(L) = \dim(U)$,
(4) L is negative semidefinite $\iff \lambda_i \leq 0$ for all $i \iff$ $\text{index}\,(L) = 0$, *and*
(5) L is indefinite $\iff \lambda_i > 0$ and $\lambda_j < 0$ for some $i, j \iff$ $0 < \text{index}\,(L) < \text{rank}(L)$.

Proof: We prove Part (1). Note that A is symmetric, and hence orthogonally diagonalizable. Choose an orthogonal matrix $P \in O_n(\mathbb{R})$ such that $P^T AP = D = \text{diag}(\lambda_1, \cdots, \lambda_n)$. Then L is positive definite $\iff$ D is positive definite $\iff$ $x^T Dx > 0$ for all $0 \neq x \in \mathbb{R}^n$ $\iff \sum_{i=1}^{n} \lambda_i {x_i}^2 > 0$ for all $0 \neq x \in \mathbb{R}^n \iff \lambda_i > 0$ for all i.

1.22 Theorem: *(The Characterization of Definiteness by Determinants) Let U be an n-dimensional vector space over $\mathbb{R}$ and let $L : U \times U \to \mathbb{R}$ be a symmetric bilinear form. Let $\mathcal{A}$ be a basis for U and let $A = [L]_{\mathcal{A}} \in M_n(\mathbb{R})$. For $1 \le k \le n$, let $A^{k\times k}$ be the upper-left $k \times k$ submatrix of A. Then*
(1) L is positive definite $\iff \det\big(A^{k\times k}\big) > 0$ for all k, and
(2) L is negative definite $\iff (-1)^k \det\big(A^{k\times k}\big) > 0$ for all k.

Proof: Suppose first that L is positive definite. Then A is positive definite, so we have $x^T A x > 0$ for all $0 \ne x \in \mathbb{R}^n$. Let $1 \le k \le n$. Note that $x^T A^{k\times k} x = \begin{pmatrix} x \\ 0 \end{pmatrix}^T A \begin{pmatrix} x \\ 0 \end{pmatrix} > 0$ for all $0 \ne x \in \mathbb{R}^k$ and so $A^{k\times k}$ is positive definite. Since $A^{k\times k}$ is positive definite, its eigenvalues are all positive and hence $\det\big(A^{k\times k}\big) > 0$ (since the determinant of a square matrix is equal to the product of its eigenvalues).

Conversely, suppose that $\det\big(A^{k\times k}\big) > 0$ for $1 \le k \le n$. Consider what happens when we apply the row and column operation algorithm (from Theorem 1.12) to diagonalize the symmetric matrix A. Since $A_{1,1} = \det\big(A^{1\times 1}\big) > 0$, we begin by using the row and column operations

$$R_i \mapsto R_i - \frac{A_{i,1}}{A_{1,1}} R_1 \text{ and } C_i \mapsto C_i - \frac{A_{1,i}}{A_{1,1}} C_1$$

to eliminate the other entries on the first row and column. This puts the matrix A into the form

$$\begin{pmatrix} A_{1,1} & 0 \\ 0 & B \end{pmatrix}$$

for some symmetric matrix B. Notice that for $1 \le k < n$, the same row and column operations convert $A^{(k+1)\times(k+1)}$ to the matrix

$$\begin{pmatrix} A_{1,1} & 0 \\ 0 & B^{k\times k} \end{pmatrix}$$

and these operations do not change the determinant so we have

$$\det\big(A^{(k+1)\times(k+1)}\big) = A_{1,1} \det\big(B^{k\times k}\big)$$

and so we have $\det\big(B^{k\times k}\big) > 0$ for $1 \le k < n$. Thus repeating this procedure eventually converts A to a diagonal matrix whose diagonal entries are all positive. It follows that $\operatorname{index}(L) = n$ and hence L is positive definite. This proves Part (1).

Finally, note that Part (2) follows immediately from Part (1) because

$$\begin{aligned} L \text{ is negative definite } &\iff -L \text{ is positive definite } \iff \det\big(-A^{k\times k}\big) > 0 \text{ for all k} \\ &\iff (-1)^k \det\big(A^{k\times k}\big) > 0 \text{ for all k.} \end{aligned}$$