MATH 247 Calculus 3, Solutions to Assignment 5

1: (a) Find $\iint_D y\,e^x\,dA$ where D is the region in $\mathbb{R}^2$ bounded by $y=0$, $y=x$ and $x+y=2$.

Solution: We have

$$\iint_D y\,e^x\,dA = \int_{y=0}^{1}\int_{x=y}^{2-y} y\,e^x\,dx\,dy = \int_{y=0}^{1}\Big[y\,e^x\Big]_{x=y}^{2-y} dy = \textstyle\int_{y=0}^{1} y\,e^{2-y} - ye^y\ dy$$
$$= \Big[-(y+1)e^{2-y} - (y-1)e^y\Big]_{y=0}^{1} = e^2 - 2e - 1.$$

(b) Find $\iint_D \frac{x}{\sqrt{1+x^2+y^2}}\,dA$ where $D = \big\{(x,y)\,\big|\,0\le x\le 2\,,\ 0\le y\le \frac12 x^2\big\}$.

Solution: Note that we can also write $D = \big\{(x,y)\,\big|\,0\le y\le 2\,,\ \sqrt{2y}\le x\le 2\big\}$ and so

$$\iint_D \frac{x}{\sqrt{1+x^2+y^2}}\,dA = \int_{y=0}^{2}\int_{x=\sqrt{2y}}^{2}\frac{x}{\sqrt{1+x^2+y^2}}\,dx\,dy = \int_{y=0}^{2}\Big[\sqrt{1+x^2+y^2}\Big]_{x=\sqrt{2y}}^{2} dx$$
$$= \int_{y=0}^{2}\sqrt{5+y^2} - \sqrt{1+2y+y^2}\ dy = \int_{y=0}^{2}\sqrt{5+y^2} - (1+y)\ dy\,.$$

Using the substitution $\sqrt5\tan\theta = y$ so that $\sqrt5\sec\theta = \sqrt{5+y^2}$ and $\sqrt5\sec^2\theta\,d\theta = dy$, we have

$$\int\sqrt{5+y^2}\,dy = \int 5\sec^3\theta\,d\theta = \tfrac52\big(\sec\theta\tan\theta + \ln(\sec\theta+\tan\theta)\big) + c$$
$$= \tfrac52\Big(\tfrac{y\sqrt{5+y^2}}{5} + \ln\big(\tfrac{y+\sqrt{5+y^2}}{5}\big)\Big) + c = \tfrac12\,y\sqrt{5+y^2} + \tfrac52\ln\big(5+\sqrt{5+y^2}\big) + d$$

(where $d = c - \frac52\ln 5$). Thus

$$\iint_D \frac{x}{\sqrt{1+x^2+y^2}}\,dA = \Big[\tfrac12\,y\sqrt{5+y^2} + \tfrac52\ln\big(5+\sqrt{5+y^2}\big) - y - \tfrac12 y^2\Big]_{y=0}^{2}$$
$$= \big(\tfrac12\cdot 6 + \tfrac52\ln(2+3) - 2 - 2\big) - \big(\tfrac52\ln\sqrt5\big) = \tfrac54\ln 5 - 1.$$

(c) Find $\iiint_D z\ dV$ where $D = \big\{(x,y,z)\,\big|\,0\le x\,,\ 0\le y\le\sqrt{x^2+z^2}\,,\ 0\le z\le\sqrt{1-x^2}\big\}$.

Solution: We have

$$\iiint_D z\ dV = \int_{x=0}^{1}\int_{z=0}^{\sqrt{1-x^2}}\int_{y=0}^{\sqrt{x^2+z^2}} z\ dy\,dz\,dx = \int_{x=0}^{1}\int_{z=0}^{\sqrt{1-x^2}} z\sqrt{x^2+z^2}\ dz\,dx$$
$$= \int_{x=0}^{1}\Big[\tfrac13(x^2+z^2)^{3/2}\Big]_{z=0}^{\sqrt{1-x^2}} dx = \int_{x=0}^{1}\tfrac13 - \tfrac13 x^3\ dx = \tfrac13 - \tfrac1{12} = \tfrac14.$$

2: (a) Find $\displaystyle\iint_D \cos(3x^2+y^2)\,dA$ where $D=\left\{(x,y)\,\middle|\,x^2+\frac{1}{3}\,y^2\leq 1\right\}$.

Solution: The change of variables map $(x,y)=g(r,\theta)=\big(r\cos\theta\,,\ \sqrt{3}\,r\,\sin\theta\big)$ sends $C=[0,1]\times[0,2\pi]$ to the given region D, and we have $Dg=\begin{pmatrix}\cos\theta & -r\sin\theta\\ \sqrt{3}\sin\theta & \sqrt{3}r\cos\theta\end{pmatrix}$ so that $\det Dg=\sqrt{3}r$, and when $(x,y)=g(r,\theta)$ we have $3x^2+y^2=3r^2\cos^2\theta+3r^2\sin^2\theta=3r^2$, and so

$$\begin{aligned}\iint\cos(3x^2+y^2)\,dA&=\int_{r=0}^{1}\int_{\theta=0}^{2\pi}\cos(3r^2)\sqrt{3}r\;d\theta\,dr=\int_{r=0}^{1}2\pi\sqrt{3}\,r\cos(3r^2)\,dr\\&=\left[\tfrac{\pi}{\sqrt{3}}\sin(3r^2)\right]_{r=0}^{1}=\tfrac{\pi}{\sqrt{3}}\sin 3.\end{aligned}$$

(b) Find $\displaystyle\iint_D e^{(y-x)/(y+x)}\,dA$ where D is the quadrilateral with vertices at $(1,1)$, $(2,0)$, $(4,0)$, $(2,2)$.

Solution: When $u=y+x$ and $v=y-x$ we have $y=\frac{u+v}{2}$ and $x=\frac{u-v}{2}$ and the lines $x+y=2$, $x+y=4$, $yt=0$ and $y=x$ (which form the boundary of D) are given by $u=2$, $v=4$, $u+v=0$ and $v=0$. So the change of variables map $(x,y)=g(u,v)=\left(\frac{u-v}{2},\frac{u+v}{2}\right)$ sends the set $C=\left\{(u,v)\,\middle|\,2\leq u\leq 4\,,\ -u\leq v\leq 0\right\}$ to the given region D. We have $Dg=\frac{1}{2}\left(\begin{smallmatrix}1&-1\\1&1\end{smallmatrix}\right)$ so that $\det Dg=\frac{1}{2}$, and so

$$\begin{aligned}\iint_D e^{(y-x)/(y+x)}\,dA&=\int_{u=2}^{4}\int_{v=-u}^{0}e^{v/u}\,dv\,du=\int_{u=2}^{4}\left[\tfrac{u}{2}\,e^{v/u}\right]_{v=-u}^{0}du\\&=\int_{u=2}^{4}\tfrac{u}{2}\big(1-\tfrac{1}{e}\big)\,du=\left[\tfrac{u^2}{4}\big(1-\tfrac{1}{e}\big)\right]_{u=2}^{4}=3\big(1-\tfrac{1}{e}\big).\end{aligned}$$

(c) Find $\displaystyle\iiint_D (x-y)z\,dV$ where $D=\left\{(x,y,z)\,\middle|\,x^2+y^2+z^2\leq 4\,,\ z\geq\sqrt{x^2+y^2}\,,\ x\geq 0\right\}$.

Solution: The region D can be described in spherical coordinates by $0\leq r\leq 2$, $0\leq\varphi\leq\frac{\pi}{4}$, and $0\leq\theta\leq\pi$. In other words, the spherical coordinates map sends the set $C=\left\{(r,\varphi,\theta)\,\middle|\,0\leq r\leq 2\,,\ 0\leq\varphi\leq\frac{\pi}{4}\,,\ 0\leq\theta\leq\pi\right\}$ to the given region D. Thus we have

$$\begin{aligned}\iiint_D(x-y)z\,dV&=\int_{r=0}^{2}\int_{\varphi=0}^{\frac{\pi}{4}}\int_{\theta=0}^{\pi}\big(r\sin\varphi\cos\theta-r\sin\varphi\sin\theta\big)(r\cos\varphi)\,r^2\sin\varphi\;d\theta\,d\varphi\,dr\\&=\int_{r=0}^{2}\int_{\varphi=0}^{\frac{\pi}{4}}\int_{\theta=0}^{\pi}r^4\cdot\sin^2\varphi\cos\varphi\cdot(\cos\theta-\sin\theta)\;d\theta\,d\varphi\,dr\\&=\left(\int_{r=0}^{2}r^4\,dr\right)\left(\int_{\varphi=0}^{\frac{\pi}{4}}\sin^2\varphi\cos\varphi\;d\varphi\right)\left(\int_{\theta=0}^{\pi}(\cos\theta-\sin\theta)\;d\theta\right)\\&=\left[\tfrac{1}{5}\,r^5\right]_{r=0}^{2}\left[\tfrac{1}{3}\sin^3\varphi\right]_{\varphi=0}^{\frac{\pi}{4}}\Big[\sin\theta+\cos\theta\Big]_{\theta=0}^{\pi}=\tfrac{32}{5}\cdot\tfrac{1}{6\sqrt{2}}\cdot 2=\tfrac{16\sqrt{2}}{15}.\end{aligned}$$

3: (a) Find the total charge in the region $D=\left\{(x,y,z)\Big|\sqrt{\frac{1}{3}(x^2+y^2)}\le z\le\sqrt{4-x^2-y^2}\right\}$ where the charge density (charge per unit volume) is given by $f(x,y,z)=x^2$.

Solution: We use spherical coordinates $(x,y,z)=\big(r\sin\phi\cos\theta,r\sin\phi\sin\theta,r\cos\phi\big)$. Some students will see immediately that the cone $z=\sqrt{\frac{1}{3}(x^2+y^2)}$ is given in spherical coordinates by $\phi=\frac{\pi}{3}$. If you do not see this immediately, then you can verify this algebraically as follows. We have

$$x^2+y^2=(r\sin\phi\cos\theta)^2+(r\sin\phi\sin\theta)^2=r^2\sin^2\phi\,(\cos^2\theta+\sin^2\theta)=r^2\sin^2\phi$$

and so (since $r\ge 0$ and $\sin\phi\ge 0$)

$$z=\sqrt{\tfrac{1}{3}(x^2+y^2)}\iff r\cos\phi=\sqrt{\tfrac{1}{3}r^2\sin^2\phi}=\tfrac{1}{\sqrt{3}}r\sin\phi\iff\tan\phi=\sqrt{3}\iff\phi=\tfrac{\pi}{3}.$$

Thus the region D is described in spherical coordinates by $0\le r\le 2$, $0\le\phi\le\frac{\pi}{3}$ and $0\le\theta\le 2\pi$. Thus the total charge is

$$\begin{aligned}Q&=\iiint_D x^2\,dV=\int_{r=0}^2\int_{\phi=0}^{\pi/3}\int_{\theta=0}^{2\pi}(r\sin\phi\cos\theta)^2\cdot r^2\sin\phi\,d\theta\,d\phi\,dr\\&=\int_{r=0}^2\int_{\phi=0}^{\pi/3}\int_{\theta=0}^{2\pi}r^4\sin^3\phi\cos^2\theta\,d\theta\,d\phi\,dr=\int_{r=0}^2\int_{\phi=0}^{\pi/3}\pi\,r^4\sin^3\phi\,d\phi\,dr\\&=\int_{r=0}^2\int_{\phi=0}^{\pi/3}\pi\,r^4(1-\cos^2\phi)\sin\phi\,d\phi\,dr=\int_{r=0}^2\pi\,r^4\Big[-\cos\phi+\tfrac{1}{3}\cos^3\phi\Big]_{\phi=0}^{\pi/3}dr\\&=\int_{r=0}^2\pi\,r^4\Big(\big(-\tfrac{1}{2}+\tfrac{1}{24}\big)-\big(-1+\tfrac{1}{3}\big)\Big)\,dr=\int_{r=0}^2\pi\,r^4\cdot\tfrac{-12+1+24-8}{24}\,dr\\&=\int_{r=0}^2\tfrac{5\pi}{24}\,r^4\,dr=\Big[\tfrac{\pi}{24}\,r^5\Big]_{r=0}^2=\tfrac{32\pi}{24}=\tfrac{4\pi}{3}.\end{aligned}$$

(b) Read Definition 7.13 and Note 7.14. Find the mass of the sphere $x^2+y^2+z^2=1$ when the density (mass per unit area) is given by $f(x,y,z)=3-z$ (this is Exercise 7.17).

Solution: The sphere is the image of the map $\sigma:[0,\pi]\times[0,2\pi]\to\mathbb{R}^3$ given by $(x,y,z)=\sigma(\varphi,\theta)=\big(\sin\varphi\cos\theta\,,\,\sin\varphi\sin\theta\,,\,\cos\varphi\big)$. We have

$$D\sigma=\big(\sigma_\varphi,\sigma_\theta\big)=\begin{pmatrix}\cos\varphi\cos\theta&-\sin\varphi\sin\theta\\\cos\varphi\sin\theta&\sin\varphi\cos\theta\\-\sin\varphi&0\end{pmatrix}\quad\text{and}\quad\sigma_\varphi\times\sigma_\theta=\begin{pmatrix}\sin^2\varphi\\\sin^2\varphi\sin\theta\\\sin\varphi\cos\varphi\end{pmatrix}$$

and hence $\big|\sigma_\varphi\times\sigma_\theta\big|=|\sin\varphi|\sqrt{\sin^2\varphi\cos^2\theta+\sin^2\varphi\sin^2\theta+\cos^2\varphi}=|\sin\varphi|=\sin\varphi$ (since $0\le\varphi\le\pi$). Thus the mass is given by

$$M=\int_{\varphi=0}^{\pi}\int_{\theta=0}^{2\pi}(3-\cos\varphi)\sin\varphi\,d\theta\,d\varphi=2\pi\int_{\varphi=0}^{\pi}3\sin\varphi-\sin\varphi\cos\varphi\,d\varphi=12\pi.$$

(c) Find the mass of the curve of intersection of the parabolic sheet $z=x^2$ with the paraboloid $z=2-x^2-2y^2$ when the density (mass per unit length) is given by $f(x,y,z)=|xy|$ (this is Exercise 7.18).

Solution: Let us find a parametric equation for the curve C of intersection. To get $z=x^2$ and $z=2-x^2-2y^2$, we need $x^2=2-x^2-2y^2$, that is $x^2+y^2=1$. Thus we can write $(x,y)=(\cos t,\sin t)$ with $t\in[0,2\pi]$. We also need $z=x^2$, so the curve C is given parametrically by $(x,y,z)=\alpha(t)=\big(\cos t,\sin t,\cos^2 t\big)$. We have $\alpha'(t)=\big(-\sin t,\cos t,-2\sin t\cos t\big)$ and $\big|\alpha'(t)\big|=\sqrt{\sin^2 t+\cos^2 t+4\sin^2 t\cos^2 t}=\sqrt{1+\sin^2(2t)}$. Using the substitution $u=\cos(2t)$ so $du=-2\sin(2t)\,dt$, the mass is given by

$$\begin{aligned}M&=\int_{t=0}^{2\pi}\big|\cos t\sin t\big|\sqrt{1+\sin^2(2t)}\,dt=\int_{t=0}^{2\pi}\big|\tfrac{1}{2}\sin(2t)\big|\sqrt{1+\sin^2(2t)}\,dt=8\int_{t=0}^{\frac{\pi}{2}}\tfrac{1}{2}\sin(2t)\sqrt{1+\sin^2(2t)}\,dt\\&=\int_{t=0}^{\pi/2}4\sin(2t)\sqrt{2-\cos^2(2t)}\,dt=\int_{u=1}^{0}-2\sqrt{2-u^2}\,du=2\int_{u=0}^{1}\sqrt{2-u^2}=\tfrac{\pi}{2}+1\end{aligned}$$

(the final value was obtained by noticing that the integral $\int_0^1\sqrt{2-u^2}\,du$ measures the area of a region consisting of one eighth of the disc of radius $\sqrt{2}$ along with a triangle of base 1 and height 1).

4: Let $f:[a,b]\to[c,d]$ be bijective and decreasing with $f(a)=d$ and $f(b)=c$, and let $g=f^{-1}:[c,d]\to[a,b]$.

(a) Suppose f and g are differentiable and consider the volume of the solid obtained by revolving the region $a\le x\le b$, $c\le y\le f(x)$ about the x-axis. Prove (using theorems from Calculus 2) that when we calculate the volume using polar coordinates for y and z, Fubini's Theorem holds so that

$$\int_{x=a}^{b}\int_{r=c}^{f(x)}\int_{\theta=0}^{2\pi} r\,d\theta\,dr\,dx=\int_{\rho=c}^{d}\int_{\varphi=0}^{2\pi}\int_{x=a}^{g(\rho)}\rho\,dx\,d\varphi\,d\rho.$$

In other words, prove that we obtain the same value using the "discs method" or using the "shells method".

Solution: Make the substitution $y=f(x)$ and integrate by parts using $u=\pi(x-a)$ and $v=f(x)^2$ so that $du=\pi\,dx$ and $dv=2f(x)f'(x)\,dx$ to get

$$\begin{aligned}\int_{y=c}^{d}2\pi\,y\,\big(g(y)-a\big)\,dy&=\int_{x=g(c)}^{g(d)}2\pi f(x)\big(g(f(x))-a\big)\,f'(x)\,dx=\int_{x=b}^{a}2\pi\,f(x)(x-a)\,f'(x)\,dx\\&=\Big[\pi(x-a)f(x)^2\Big]_{x=b}^{a}-\int_{x=b}^{a}\pi f(x)^2\,dx=-\pi(b-a)f(b)^2+\int_{x=a}^{b}\pi f(x)^2\,dx\\&=-\pi(b-a)\,c^2+\int_{x=a}^{b}\pi f(x)^2\,dx=-\int_{x=a}^{b}\pi c^2\,dx+\int_{x=a}^{b}\pi f(x)^2\,dx=\int_{x=a}^{b}\pi\big(f(x)^2-c^2\big)\,dx.\end{aligned}$$

(b) Suppose f and g are continuous and consider the area of the region $a\le x\le b$, $c\le y\le f(x)$. Prove (using theorems from Calculus 2) that Fubini's Theorem holds, that is

$$\int_{x=a}^{b}\int_{y=c}^{f(x)}1\,dy\,dx=\int_{y=c}^{d}\int_{x=a}^{g(y)}1\,dx\,dy$$

.

Solution: We need to show that

$$\int_{x=a}^{b}f(x)\,dx-\int_{y=c}^{d}g(y)\,dy=\int_{x=a}^{b}c\,dx-\int_{y=c}^{d}a\,dy=c(b-a)-a(d-c)=bc-ad.$$

Let $\epsilon>0$ be arbitrary. Choose $\delta_1>0$ so that for every partition X of $[a,b]$ with $|X|<\delta_1$ we have $\big|S-\int_a^b f\big|<\frac12\,\epsilon$ for every Riemann sum S for f on X, and choose $\delta_2>0$ such that for every partition Y of $[c,d]$ with $|Y|<\delta_2$ we have $\big|S-\int_c^d g\big|<\frac12\,\epsilon$ for every Riemann sum S for g on Y. Choose a partition X_0 of $[a,b]$ with $|X_0|<\delta_1$ and choose a partition Y_0 of $[c,d]$ with $|Y_0|<\delta_2$. Let $X=X_0\cup g(Y_0)$ and let $Y=Y_0\cup f(X_0)$. Then we have $|X|<\delta_1$ and $|Y|<\delta_2$. Write $X=\{x_0,x_1,\cdots,x_n\}$, whith the x_k in increasing order as usual, and note that, since f is decreasing, we have $Y=\{y_0,y_1,\cdots,y_n\}$ where $y_\ell=f(x_{n-\ell})$ for all ℓ. Since f and g are decreasing, the lower Riemann sums are equal to the sums using the right endpoints. Making the substitution $k=n-\ell$ in one of the sums below and $k=n-\ell+1$ in another, we have

$$\begin{aligned}L(f,X)-L(g,Y)&=\sum_{k=1}^{n}f(x_k)(x_k-x_{k-1}-\sum_{\ell=1}^{n}g(y_\ell)(y_\ell-y_{\ell-1})\\&=\sum_{k=1}^{n}f(x_k)(x_k-x_{k-1})-\sum_{\ell=1}^{n}x_{n-\ell}\big(f(x_{n-\ell})-f(x_{n-\ell+1})\big)\\&=\sum_{k=1}^{n}x_kf(x_k)-\sum_{k=1}^{n}x_{k-1}f(x_k)-\sum_{\ell=1}^{n}x_{n-\ell}f(x_{n-\ell})+\sum_{\ell=1}^{n}x_{n-\ell}f(x_{n-\ell+1})\\&=\sum_{k=1}^{n}x_kf(x_k)-\sum_{k=1}^{n}x_{k-1}f(x_k)-\sum_{k=0}^{n-1}x_kf(x_k)+\sum_{k=1}^{n}x_{k-1}f(x_k)\\&=x_nf(x_n)-x_0f(x_0)x_0\;=\;bc-ad.\end{aligned}$$

By the Triangle Inequality

$$\begin{aligned}\left|\left(\int_a^b f(x)\,dx-\int_b^c g(y)\,dy\right)-(bc-ad)\right|&=\left|\int_a^b f(x)\,dx-\int_b^c g(y)\,dy-L(f,X)+L(g,Y)\right|\\&\le\left|\int_a^b f(x)\,dx-L(f,X)\right|+\left|\int_b^c g(y)\,dy-L(g,Y)\right|<\tfrac{\epsilon}{2}+\tfrac{\epsilon}{2}=\epsilon.\end{aligned}$$

Since $\epsilon>0$ was arbitrary, it follows that $\displaystyle\int_a^b f(x)\,dx-\int_c^d g(y)\,dy=bc-ad$, as required.