MATH 247 Calculus 3, Solutions to Assignment 4

1: (a) Define $f:\mathbb{R}^2\to\mathbb{R}$ by $f(0,0)=0$ and $f(x,y)=\frac{x^3-xy^2}{x^2+y^2}$ for $(x,y)\neq(0,0)$. Determine whether f is differentiable at $(0,0)$.

Solution: We claim that f is not differentiable at $(0,0)$. When $\alpha(t)=(t,0)$ and $g(t)=f(\alpha(t))=t$, we have $\frac{\partial f}{\partial x}(0,0)=g'(0)=1$. When $\beta(t)=(0,t)$ and $h(t)=f(\beta(t))=0$, we have $\frac{\partial f}{\partial y}(0,0)=h'(0)=0$. When $\gamma(t)=(t,t)$ and $k(t)=f(\gamma(t))=0$, if f was differentiable at $(0,0)$, then by the Chain Rule we would have $k'(0)=Df(0,0)\gamma'(0)=(1\ 0)\binom{1}{1}=1$, but instead we have $k'(0)=0$.

(b) Suppose $f:U\subseteq\mathbb{R}^n\to\mathbb{R}$ is differentiable and f has a local maximum at $a\in U$. Show that $Df(a)=O$ (this is Exercise 6.15 in the lecture notes).

Solution: Suppose, for a contradiction, that $Df(a)\neq O$. Choose $0\neq u\in\mathbb{R}^n$ such that $Df(a)u\neq 0$. By replacing u by $-u$ if necessary, we may assume that $Df(a)u=c>0$. Let $\alpha(t)=a+tu$, choose $\delta_1>0$ small enough so that $\alpha(t)\in U$ for all $|t|<\delta_1$, and let $g(t)=f(\alpha(t))$ for $|t|<\delta_1$. By the Chain Rule we have $g'(t)=Df(\alpha(t))\alpha'(t)$ so that, in particular, $g'(0)=Df(a)u=c>0$. Since $c=g'(0)=\lim\limits_{t\to 0}\frac{g(t)-g(0)}{t}$, we can choose δ with $0<\delta<\delta_1$ such that when $0<|t|<\delta$ we have $\left|\frac{g(t)-g(0)}{t}-c\right|<\frac{c}{2}$, and hence $\frac{c}{2}<\frac{g(t)-g(0)}{t}<\frac{3c}{2}$. For $0<t<\delta$ we have $g(t)-g(0)>\frac{ct}{2}>0$ so that $g(t)>g(0)$. Thus $f(a+tu)>f(a)$ for all $0<t<\delta$, and so f does not have a local maximum at a.

(c) Let $f:U\subseteq\mathbb{R}^n\to\mathbb{R}^m$. Suppose the partial derivatives $\frac{\partial f_k}{\partial x_\ell}(x)$ exist and are bounded in U. Prove that f is continuous.

Solution: We imitate the proof of Theorem 5.13. Let $\epsilon>0$. Choose $M\geq 0$ so that $\left|\frac{\partial f_k}{\partial x_\ell}(x)\right|\leq M$ for all indices k,ℓ and all $x\in U$ and choose δ with $0<\delta<\frac{\epsilon}{Mnm}$ so that $B(a,\delta)\subseteq U$. Let $x\in B(a,\delta)$. For $0\leq\ell\leq n$, let $u_\ell=(x_1,\cdots,x_\ell,a_{\ell+1},\cdots,a_n)$, with $u_0=a$ and $u_n=x$, and note that each $u_\ell\in B(a,\delta)$. For $1\leq\ell\leq n$, let $\alpha_\ell(t)=(x_1,\cdots,x_{\ell-1},t,a_{\ell+1},\cdots,a_n)$ for t between a_ℓ and x_ℓ. For $1\leq k\leq m$ and $1\leq\ell\leq n$, let $g_{k,\ell}(t)=f_k\big(\alpha_\ell(t)\big)$ so that $g'_{k,\ell}(t)=\frac{\partial f_k}{\partial x_\ell}\big(\alpha_\ell(t)\big)$. By the Mean Value Theorem, we can choose $s_{k,\ell}$ between a_ℓ and x_ℓ so $g'_{k,\ell}(s_{k,\ell})(x_\ell-a_\ell)=g_{k,\ell}(x_\ell)-g_{k,\ell}(a_\ell)$ or, equivalently, so $\frac{\partial f_k}{\partial x_\ell}\big(\alpha_\ell(s_{k,\ell})\big)(x_\ell-a_\ell)=f_k(u_\ell)-f_k(u_{\ell-1})$. Then

$$f_k(x)-f_k(a)=f_k(u_n)-f_k(u_0)=\sum_{\ell=1}^n\big(f_k(u_\ell)-f_k(u_{\ell-1})\big)=\sum_{\ell=1}^n\frac{\partial f_k}{\partial x_\ell}\big(\alpha_\ell(s_{k,\ell})\big)(x_\ell-a_\ell),$$

so that $\big|f_k(x)-f_k(a)\big|\leq M\sum\limits_{\ell=1}^n\big|x_\ell-a_\ell\big|\leq Mn\,|x-a|$. Thus

$$\big|f(x)-f(a)\big|=\Big(\sum_{k=1}^m\big|f_k(x)-f_k(a)\big|^2\Big)^{1/2}\leq\Big(\sum_{k=1}^m n^2M^2|x-a|^2\Big)^{1/2}=Mnm\,|x-a|<Mnm\,\delta<\epsilon.$$

2: (a) Let $(u,v)=f(x,y)=\left(x\ln(y-x^4),\left(2+\frac{y}{x}\right)^{3/2}\right)$. Explain why f is locally invertible in a neighbourhood of $(1,2)$ and find the linearization of its inverse at $(0,8)$.

Solution: Note that $f(1,2)=(0,8)$. Also

$$DF(x,y)=\begin{pmatrix}u_x & u_y\\ v_x & v_y\end{pmatrix}=\begin{pmatrix}\ln(y-x^4)-\frac{4x^2}{y-x^4} & \frac{x}{y-x^4}\\ -\frac{3y}{2x^2}\left(2+\frac{y}{x}\right)^{1/2} & \frac{3}{2x}\left(2+\frac{y}{x}\right)^{1/2}\end{pmatrix}\text{ , so } DF(1,2)=\begin{pmatrix}-4 & 1\\ -6 & 3\end{pmatrix}.$$

F is locally invertible near $(1,2)$ because the matrix $DF(1,2)$ is invertible, and the partial derivatives u_x, u_y, v_x and v_y are all continuous near $(1,2)$. Since $F(1,2)=(0,8)$ we have $F^{-1}(0,8)=(1,2)$, and we have

$$DF^{-1}(0,8)=F(1,2)^{-1}=\begin{pmatrix}-4 & 1\\ -6 & 3\end{pmatrix}^{-1}=\tfrac{1}{6}\begin{pmatrix}3 & -1\\ 6 & -4\end{pmatrix}$$

and so the linearization of F^{-1} at $(0,8)$ is

$$L_{(0,8)}F^{-1}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}1\\ 2\end{pmatrix}+\tfrac{1}{6}\begin{pmatrix}3 & -1\\ 6 & -4\end{pmatrix}\begin{pmatrix}x-0\\ y-8\end{pmatrix}.$$

(b) Define $f:\mathbb{R}^2\to\mathbb{R}$ by $f(x,y)=2x^3-3x^2+2y^3+3y^2$ and let $C=\text{Null}(f)$. Use the Implicit Function Theorem to find all the points on C at which C is locally equal to the graph of a function $y=g(x)$, or locally equal to the graph of a function $x=h(y)$.

Solution: By the Implicit Function Theorem, C is locally equal to the graph of a smooth function $y=g(x)$ at all points on C except (possibly) where $\frac{\partial f}{\partial y}=0$. We have $\frac{\partial f}{\partial y}=6y^2+6y=6y(y+1)$ and so $\frac{\partial f}{\partial y}=0\iff y=0$ or $y=-1$. For $(x,y)\in C$ we have $(x+y)(2x^2-2xy+2y^2-3x+3y)=0$ so

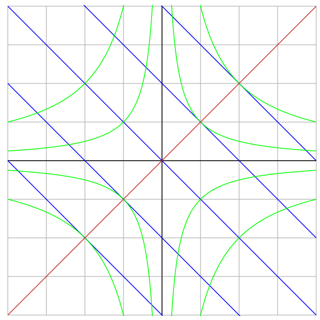
$$\begin{aligned}y=0&\implies x(2x^2-3x)=0\implies x^2(2x-3)=0\implies x=0\text{ or }\tfrac{3}{2},\\ y=-1&\implies (x-1)(2x^2-x-1)=0\implies (x-1)^2(2x+1)=0\implies x=1\text{ or }-\tfrac{1}{2}.\end{aligned}$$

Thus C is locally equal to the graph of a smooth function $y=g(x)$ except (possibly) at the points $(0,0)$, $\left(\frac{3}{2},0\right)$, $(1,-1)$ and $\left(-\frac{1}{2},-1\right)$. A similar calculation shows that $\frac{\partial f}{\partial x}=6x(x-1)$ and that for $(x,y)\in C$ we have $x=0\implies y=0$ or $-\frac{3}{2}$ and $x=1\implies y=-1$ or $\frac{1}{2}$, and so C is locally equal to the graph of a smooth function $x=h(y)$ except (possibly) at each of the points $(0,0)$, $\left(0,-\frac{3}{2}\right)$, $(1,-1)$ and $\left(1,\frac{1}{2}\right)$.

3: Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $(u, v) = f(x, y) = \big(x + y, xy\big)$.

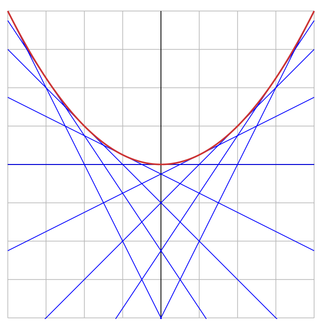
(a) Sketch the level sets $u = 0, \pm 2, \pm 4$ and the level sets $v = 0, \pm 1, \pm 4$ (all on the same grid).

Solution: The level set $u = a$ is the line $x + y = a$ and the level set $v = b$ is the hyperbola $xy = b$ (when $b = 0$ we obtain the degenerate hyperbola $xy = 0$, which is the union of the two coordinate axes). The lines $x + y = a$ for $a = 0, \pm 2, \pm 4$ are whown below in blue, and the hyperbolas $xy = b$ for $b = \pm 1, \pm 4$ are shown in green.



(b) Sketch the image under f of each of the lines $x = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 2$ (all on the same grid).

Solution: The level set $x = c$ is given parametrically by $(x, y) = (c, t)$ and it is mapped under f to the curve $(u, v) = f(c, t) = (c+t, ct)$. When $u = c+t$ and $v = ct$ we have $cu = c^2 + ct = c^2 + v$ and so the image of the curve $x = c$ is the line $cu = c^2 + v$. The lines $v = cu - c^2$ for $c = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 2$ are shown below in blue.



(c) Let $A = \big\{(x, y)\big| \det\big(f'(x, y)\big) = 0\big\}$ and $B = f(A)$. Find a function $y = y(x)$ whose graph is A and a function $v = v(u)$ whose graph is B. Add A to your sketch in Part (a) and add B to your sketch in Part (b).

Solution: We have

$$f'(x, y) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ y & x \end{pmatrix}$$

and so $A = \big\{(x, y)\big| \det f'(x, y) = 0\big\} = \big\{(x, y)\big| x - y = 0\big\}$ which is equal to the line $y = x$, shown in orange on the plot in Part (a). The line $y = x$ is given parametrically by $(x, y) = (t, t)$ and it is sent by f to the curve $(u, v) = f(t, t) = (2t, t^2)$. When $u = 2t$ and $v = t^2$ we have $4v = 4t^2 = (2t)^2 = u^2$ and so the image of the line $y = x$ under f is the parabola $4v = u^2$, which is shown in orange on the Plot in Part (b).

(d) Show that for $U = \{(x,y) | y < x\}$ and $V = \{(u,v) | 4v < u^2\}$ the map $f : U \to V$ is invertible and find a formula for $g = f^{-1} : V \to U$.

Solution: Note that

$$\begin{aligned}(u,v) = (x+y, xy) &\implies \big(u = x+y \text{ and } v = xy\big) \implies \big(y = u - x \text{ and } v = xy = x(u-x)\big) \\ &\implies x^2 - ux + v = 0 \implies x = \tfrac{u \pm \sqrt{u^2-4v}}{2}\end{aligned}$$

and similarly

$$\begin{aligned}(u,v) = (x+y, xy) &\implies \big(u = x+y \text{ and } v = xy\big) \implies \big(x = u - y \text{ and } v = xy = (u-y)y\big) \\ &\implies y^2 - uy + v = 0 \implies y = \tfrac{u \pm \sqrt{u^2-4v}}{2}.\end{aligned}$$

To get $y < x$ we need $x = \frac{u+\sqrt{u^2-4v}}{2}$ and $y = \frac{u-\sqrt{u^2-4v}}{2}$, and so we define $g : V \to \mathbb{R}^2$ by

$$(x,y) = g(u,v) = \Big(\tfrac{u+\sqrt{u^2-4v}}{2}, \tfrac{u-\sqrt{u^2-4v}}{2}\Big).$$

Note the $g(u,v)$ is well defined when $4v < u$, that is when $(u,v) \in V$. Let us verify that g is indeed the inverse of the restriction of f to U. For $y < x$ we have

$$\begin{aligned}g\big(f(x,y)\big) = g(x+y, xy) &= \Big(\tfrac{(x+y)+\sqrt{(x+y)^2-4xy}}{2}, \tfrac{(x+y)-\sqrt{(x+y)^2-4xy}}{2}\Big) \\ &= \Big(\tfrac{(x+y)+\sqrt{(x-y)^2}}{2}, \tfrac{(x+y)-\sqrt{(x-y)^2}}{2}\Big) = \Big(\tfrac{(x+y)+(x-y)}{2}, \tfrac{(x+y)-(x-y)}{2}\Big) = (x,y)\end{aligned}$$

and when $4v < u^2$ we have

$$\begin{aligned}f\big(g(u,v)\big) &= f\Big(\tfrac{u+\sqrt{u^2-4v}}{2}, \tfrac{u-\sqrt{u^2-4v}}{2}\Big) = \Big(\tfrac{u+\sqrt{u^2-4v}}{2} + \tfrac{u-\sqrt{u^2-4v}}{2}\,,\, \tfrac{u+\sqrt{u^2-4v}}{2} \cdot \tfrac{u-\sqrt{u^2-4v}}{2}\Big) \\ &= \Big(u\,,\, \tfrac{u^2-(u^2-4v)}{4}\Big) = (u,v).\end{aligned}$$

(e) Note that $f(2,1) = (3,2)$. Find $g'(3,2)$ in two ways: first use the Inverse Function Theorem, then use your formula for g from Part (d).

Solution: Using the Inverse Function Theorem, we have

$$f'(x,y) = \begin{pmatrix} 1 & 1 \\ y & x \end{pmatrix} \ ,\ f'(2,1) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \ ,\ g'(3,2) = f'(2,1)^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

Using the formula $g(u,v) = \frac{1}{2}\big(u + \sqrt{u^2-4v}\,,\, u - \sqrt{u^2-4v}\big)$ we have

$$g'(u,v) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 + \frac{u}{\sqrt{u^2-4v}} & \frac{-2}{\sqrt{u^2-4v}} \\ 1 - \frac{u}{\sqrt{u^2-4v}} & \frac{2}{\sqrt{u^2-4v}} \end{pmatrix} \ ,\ g'(3,2) = \frac{1}{2}\begin{pmatrix} 1+3 & -2 \\ 1-3 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

4: (a) Let $U = \left\{(x,y) \in \mathbb{R}^2 \middle| x^2 > y^2\right\}$. Find the 2^{nd} Taylor polynomial of the map $f : U \to \mathbb{R}$ given by $f(x,y) = \sqrt{x^2 - y^2}$ at the point $(5,4)$.

Solution: We have $\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2-y^2}}$, $\frac{\partial f}{\partial y} = \frac{-y}{\sqrt{x^2-y^2}}$, $\frac{\partial^2 f}{\partial x^2} = \frac{\sqrt{x^2-y^2} - \frac{x^2}{\sqrt{x^2-y^2}}}{x^2-y^2} = \frac{-y^2}{(x^2-y^2)^{3/2}}$, $\frac{\partial^2 f}{\partial x \partial y} = \frac{xy}{(x^2-yy^2)^{3/2}}$, and $\frac{\partial^2 f}{\partial y^2} = \frac{-\sqrt{x^2-y^2} - \frac{y^2}{\sqrt{x^2-y^2}}}{x^2-y^2} = \frac{-x^2}{(x^2-y^2)^{3/2}}$, so that $f(5,4) = 3$, $\frac{\partial f}{\partial x}(5,4) = \frac{5}{3}$, $\frac{\partial f}{\partial y}(5,4) = -\frac{4}{3}$, $\frac{\partial^2 f}{\partial x^2}(5,4) = -\frac{16}{27}$, $\frac{\partial^2 f}{\partial x \partial y}(5,4) = \frac{20}{27}$ and $\frac{\partial^2 f}{\partial y^2}(5,4) = -\frac{25}{27}$, and hence the 2^{nd} Taylor polynomial of f at $(5,4)$ is

$$T(x,y) = 3 + \tfrac{5}{3}(x-5)_{-}\tfrac{4}{3}(y-4) - \tfrac{8}{27}(x-5)^2 + \tfrac{20}{27}(x-5)(y-4) - \tfrac{25}{54}(y-4)^2.$$

(b) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = 2x + x^2 + y^2 - xy^2$. Find the absolute maximum and minimum values of $f(x,y)$ in the region $D = \left\{(x,y) \middle| y^2 - 4 \le 2x \le 4\right\}$.

Solution: Since D is compact, f does attain its maximum and minimum values in D, and these values are either attained in D^o or on ∂D. If f attains a maximum or minimum value in D^o then it must do so at a critical point. Note that $Df(x,y) = \big(2 + 2x - y^2\,,\, 2y - 2xy\big)$. To have $Df(x,y) = (0,0)$ we need $0 = 2y - 2xy = 2y(1-x)$ so that either $y = 0$ or $x = 1$, and we need $2 + 2x = y^2$. When $y = 0$ we have $2 + 2x = y^2 \implies 2 + 2x = 0 \implies x = -1$, and when $x = 1$ we have $2 + 2x = y^2 \implies y^2 = 4 \implies y = \pm 2$. Thus the critical points are $(-1,0)$ and $(1,\pm 2)$, which all lie in D^o, and we have $f(-1,0) = -2 + 1 = -1$ and $f(1,\pm 2) = 2 + 1 + 4 - 4 = 3$. Let us determine the maximum and minimum values on ∂D.

The region D is bounded by the parabola $y^2 - 4 = 2x$ and the line $2x = 4$ that is $x = 2$. The parabola and the line intersect when $x = 2$ and $y^2 = 2x + 4 = 8$, that is at the points $(x,y) = (2, \pm\sqrt{8})$. When $x = 2$ with $-\sqrt{8} \le y \le \sqrt{8}$, we have

$$f(x,y) = 2x + x^2 + y^2 - xy^2 = 4 + 4 + y^2 - 2y^2 = 8 - y^2$$

which has maximum value 8 when $y = 0$ and minimum value 0 when $y = \pm\sqrt{8}$. When $y^2 - 4 = 2x$ with $-\sqrt{8} \le y \le \sqrt{8}$, we have $y^2 = 2x + 4$ with $-2 \le x \le 2$ so

$$f(x,y) = 2x + x^2 + y^2 - xy^2 = 2x + x^2 + (2x+4) - x(2x+4) = 4 - x^2$$

which has maximum value 4 when $x = 0$ and minimum value 0 when $x = \pm 2$.

Taking all of the above into account, the absolute maximum value of $f(x,y)$ on D is $f(2,0) = 8$ and the absolute minimum value is $f(-1,0) = -1$.