MATH 247 Calculus 3, Assignment 4 Due: Fri Mar 22, 11:00 pm.

1: (a) Define $f:\mathbb{R}^2\to\mathbb{R}$ by $f(0,0)=0$ and $f(x,y)=\frac{x^3-xy^2}{x^2+y^2}$ for $(x,y)\neq(0,0)$. Determine whether f is differentiable at $(0,0)$.

(b) Suppose $f:U\subseteq\mathbb{R}^n\to\mathbb{R}$ is differentiable and f has a local maximum at $a\in U$. Show that $Df(a)=O$ (this is Exercise 6.15 in the lecture notes).

(c) Let $f:U\subseteq\mathbb{R}^n\to\mathbb{R}^m$. Suppose the partial derivatives $\frac{\partial f_k}{\partial x_\ell}(x)$ exist and are bounded in U. Prove that f is continuous.

2: (a) Let $(u,v)=f(x,y)=\Big(x\ln(y-x^4),\big(2+\frac{y}{x}\big)^{3/2}\Big)$. Explain why f is locally invertible in a neighbourhood of $(1,2)$ and find the linearization of its inverse at $(0,8)$.

(b) Define $f:\mathbb{R}^2\to\mathbb{R}$ by $f(x,y)=2x^3-3x^2+2y^3+3y^2$ and let $C=\text{Null}(f)$. Use the Implicit Function Theorem to find all the points on C at which C is locally equal to the graph of a function $y=g(x)$, or locally equal to the graph of a function $x=h(y)$.

3: Define $f:\mathbb{R}^2\to\mathbb{R}^2$ by $(u,v)=f(x,y)=\big(x+y,xy\big)$.

(a) Sketch the level sets $u=0,\pm2,\pm4$ and the level sets $v=0,\pm1,\pm4$ (all on the same grid).

(b) Sketch the image under f of each of the lines $x=0,\pm\frac{1}{2},\pm1,\pm\frac{3}{2},\pm2$ (all on the same grid).

(c) Let $A=\big\{(x,y)\big|\det\big(Df(x,y)\big)=0\big\}$ and $B=f(A)$. Find a function $y=y(x)$ whose graph is A and a function $v=v(u)$ whose graph is B. Add A to your sketch in Part (a) and add B to your sketch in Part (b).

(d) Show that for $U=\big\{(x,y)\big|y<x\big\}$ and $V=\big\{(u,v)\big|4v<u^2\big\}$ the map $f:U\to V$ is invertible and find a formula for $g=f^{-1}:V\to U$.

(e) Note that $f(2,1)=(3,2)$. Find $Dg(3,2)$ in two ways: first use the Inverse Function Theorem, then use your formula for g from Part (d).

4: (a) Let $U=\big\{(x,y)\in\mathbb{R}^2\big|\,x^2>y^2\big\}$. Find the 2$^{\text{nd}}$ Taylor polynomial of the map $f:U\to\mathbb{R}$, which is given by $f(x,y)=\sqrt{x^2-y^2}$, at the point $(5,4)$.

(b) Define $f:\mathbb{R}^2\to\mathbb{R}$ by $f(x,y)=2x+x^2+y^2-xy^2$. Find the absolute maximum and minimum values of $f(x,y)$ in the region $D=\big\{(x,y)\big|y^2-4\leq 2x\leq 4\big\}$.