MATH 247 Calculus 3, Solutions to Assignment 3

1: Find a parametric equation for the tangent line at $(\sqrt{3}, 2, -1)$ to the curve of intersection of the two paraboloids $z = 6 - x^2 - y^2$ and $z = x^2 + y^2 - 4y$ using the following two methods:

(a) Find a parametric equation for the curve of intersection of the two paraboloids.

Solution: To get $z = 6 - x^2 - y^2$ and $z = x^2 + y^2 - 4y$ we must have $6 - x^2 - y^2 = x^2 + y^2 - 4y$, that is $2x^2 + 2y^2 - 4y = 6$, so $x^2 + y^2 - 2y = 3$, or equivalently $x^2 + (y-1)^2 = 4$. This shows that the top view of the curve of intersection looks like the circle of radius 2 centred at $(x, y) = (0, 1)$, and so we can take

$$(x, y) = \big(2\cos t, 1 + 2\sin t\big)$$

Then we need $z = 6 - x^2 - y^2 = 6 - (2\cos t)^2 - (1 + 2\sin t)^2 = 6 - 4\cos^2 t - 1 - 4\sin t - 4\sin^2 t = 1 - 4\sin t$, so the curve of intersection is given by

$$p(t) = (x, y, z) = \big(2\cos t, 1 + 2\sin t, 1 - 4\sin t\big)$$

and $p'(t) = \big(-2\sin t, 2\cos t, -4\cos t\big)$. Note that $p\left(\frac{\pi}{6}\right) = (\sqrt{3}, 2, -1)$ and we have $p'\left(\frac{\pi}{6}\right) = (-1, \sqrt{3}, -2\sqrt{3})$, so the tangent line is given by

$$q(t) = (x, y, z) = (\sqrt{3}, 2, -1) + t\,(-1, \sqrt{3}, -2\sqrt{3})\,.$$

(b) Find the tangent plane at $(\sqrt{3}, 2, -1)$ to each of the two paraboloids, then find the line of intersection of these two planes.

Solution: For $z = 6 - x^2 - y^2$ we have $\frac{\partial z}{\partial x} = -2x$ and $\frac{\partial z}{\partial y} = -2y$ so $\frac{\partial z}{\partial x}(\sqrt{3}, 2) = -2\sqrt{3}$ and $\frac{\partial z}{\partial y}(\sqrt{3}, 2) = -4$, and so the tangent plane to the first paraboloid at $(\sqrt{3}, 2, -1)$ is given by $z = -1 - 2\sqrt{3}(x - \sqrt{3}) - 4(y - 2)$, or equivalently

$$2\sqrt{3}\,x + 4y + z = 13 \quad (1)\,.$$

For $z = x^2 + y^2 - 4y$ we have $\frac{\partial z}{\partial x} = 2x$ and $\frac{\partial z}{\partial y} = 2y - 4$, so $\frac{\partial z}{\partial x}(\sqrt{3}, 2) = 2\sqrt{3}$ and $\frac{\partial z}{\partial y}(\sqrt{3}, 2) = 0$, and so the tangent plane to the second paraboloid at $(\sqrt{3}, 2, -1)$ is given by $z = -1 + 2\sqrt{3}(x - \sqrt{3})$, or equivalently

$$2\sqrt{3}\,x - z = 7 \quad (2)\,.$$

If we set $z = s$ (an arbitrary parameter) then from equation (2) we get $2\sqrt{3}\,x = 7 + z = 7 + s$, and then from equation (1) we gat $4y = 19 - 2\sqrt{3}\,x - z = 19 - (7 + s) - s = 6 - 2s$, and so we get

$$r(s) = (x, y, z) = \left(\tfrac{7+s}{2\sqrt{3}}, \tfrac{3-s}{2}, s\right) = \left(\tfrac{7}{2\sqrt{3}}, \tfrac{3}{2}, 0\right) + s\left(\tfrac{1}{2\sqrt{3}}, -\tfrac{1}{2}, 1\right)\,.$$

(We remark that $q(t)$ and $r(s)$ give the same line since we have $r(-1) = (\sqrt{3}, 2, -1) = q(0)$ so the lines share a common point, and $-2\sqrt{3}\left(\frac{1}{2\sqrt{3}}, -\frac{1}{2}, 1\right) = (-1, \sqrt{3}, -2\sqrt{3})$ so their direction vectors are parallel).

2: Find an implicit equation (of the form $ax+by+cz=d$) for the tangent plane to the parametric surface $(x,y,z)=f(\theta,\phi)=\big(\sin 2\theta\cos\phi,\sin 2\theta\sin\phi,\cos\theta\big)$ at the point $f\big(\frac{\pi}{6},\frac{\pi}{3}\big)$ using the following two methods:

(a) Find a parametric equation for the tangent plane, then convert it to an implicit equation.

Solution: We have $f\big(\frac{\pi}{6},\frac{\pi}{3}\big)=\big(\frac{\sqrt3}{4},\frac34,\frac{\sqrt3}{2}\big)$ and we have

$$Df(\theta,\phi)=\begin{pmatrix}2\cos 2\theta\cos\phi & -\sin 2\theta\sin\phi\\ 2\cos 2\theta\sin\phi & \sin 2\theta\cos\phi\\ -\sin\theta & 0\end{pmatrix}\ \text{ so that }\ Df\big(\tfrac{\pi}{6},\tfrac{\pi}{3}\big)=\begin{pmatrix}\frac12 & -\frac34\\ \frac{\sqrt3}{2} & \frac{\sqrt3}{4}\\ -\frac12 & 0\end{pmatrix}$$

and so the tangent space is given parametrically by

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}\frac{\sqrt3}{4}\\ \frac34\\ \frac{\sqrt3}{2}\end{pmatrix}+\begin{pmatrix}\frac12 & -\frac34\\ \frac{\sqrt3}{2} & \frac{\sqrt3}{4}\\ -\frac12 & 0\end{pmatrix}\begin{pmatrix}\theta-\frac{\pi}{6}\\ \phi-\frac{\pi}{3}\end{pmatrix}=p+su+tv$$

where $p=\big(\frac{\sqrt3}{4},\frac34,\frac{\sqrt3}{2}\big)^T$, $u=\big(\frac12,\frac{\sqrt3}{2},-\frac12\big)^T$ and $v=\big(-\frac34,\frac{\sqrt3}{4},0\big)^T$, and where we let $s=\theta-\frac{\pi}{6}$ and $t=\phi-\frac{\pi}{3}$. There are several ways to convert this to an implicit equation. One way is to note that this is a plane with normal vector $u\times v=\big(\frac{\sqrt3}{8},\frac38,\frac{\sqrt3}{2}\big)^T$ so the equation is of the form $\frac{\sqrt3}{8}\,x+\frac38\,y+\frac{\sqrt3}{2}\,z=c$ for some c. We can find c by putting in $(x,y,z)=\big(\frac{\sqrt3}{4},\frac34,\frac{\sqrt3}{2}\big)$ to get $c=\frac{\sqrt3}{8}\cdot\frac{\sqrt3}{4}+\frac38\cdot\frac34+\frac{\sqrt3}{2}\cdot\frac{\sqrt3}{2}=\frac98$, so the equation is $\frac{\sqrt3}{8}\,x+\frac38\,y+\frac{\sqrt3}{2}\,z=\frac98$, or equivalently $\sqrt3\,x+3\,y+4\sqrt3\,z=9$.

(b) Find an implicit equation for the surface and use it to obtain an equation for the tangent plane.

Solution: When $(x,y,z)=\big(\sin 2\theta\cos\phi,\sin 2\theta\sin\phi,\cos\theta\big)$ we have

$$x^2+y^2=\sin^2 2\theta=4\sin^2\theta\cos^2\theta=4(1-\cos^2\theta)\cos^2\theta=4(1-z^2)z^2$$

and so the surface is given implicitly by $x^2+y^2=4z^2-4z^4$. In other words, the surface is the null set of the function $g(x,y,z)=x^2+y^2+4z^4-4z^2$. We have

$$Dg(x,y,z)=\begin{pmatrix}2x & 2y & 16z^3-8z\end{pmatrix}\ \text{ so that }\ Dg\big(\tfrac{\sqrt3}{4},\tfrac34,\tfrac{\sqrt3}{2}\big)=\begin{pmatrix}\frac{\sqrt3}{2} & \frac32 & 2\sqrt3\end{pmatrix}.$$

The equation of the tangent plane is $\frac{\sqrt3}{2}\big(x-\frac{\sqrt3}{4}\big)+\frac32\big(y-\frac34\big)+2\sqrt3\big(z-\frac{\sqrt3}{2}\big)=0$, that is $\frac{\sqrt3}{2}\,x+\frac32\,y+2\sqrt3\,z=\frac92$, or equivalently $\sqrt3\,x+3\,y+4\sqrt3\,z=9$.

3: (a) Let $(u,v)=g(z)=\big(\sqrt{z-1},5\ln z\big)$, where $z=f(x,y)=4x^2-8xy+5y^2$. Use the Chain Rule to find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ at the point $(2,1)$ (this is Exercise 4.21 in the Lecture Notes).

Solution: Write $h(x,y)=g(f(x,y))$. When $(x,y)=(2,1)$ we have $z=f(2,1)=5$ and so by the Chain Rule, we have $Dh(2,1)=Dg(5)\,Df(2,1)$, that is

$$\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}=\begin{pmatrix}\frac{du}{dz} & \frac{dv}{dz}\end{pmatrix}\begin{pmatrix}\frac{\partial z}{\partial x}\\ \frac{\partial z}{\partial y}\end{pmatrix}=\begin{pmatrix}\frac{1}{2\sqrt{z-1}} & \frac{5}{z}\end{pmatrix}\begin{pmatrix}8x-8y\\ -8x+10y\end{pmatrix}=\begin{pmatrix}\frac14 & 1\end{pmatrix}\begin{pmatrix}8\\ -6\end{pmatrix}=\begin{pmatrix}2 & -\frac32\\ 8 & -6\end{pmatrix}$$

(b) Let $(x,y)=f(r,\theta)=\big(r\cos\theta\,,\,r\sin\theta\big)$, let $z=g(x,y)$ and let $z=h(r,\theta)=g\big(f(r,\theta)\big)$. Suppose that $h(r,\theta)=r^2e^{\sqrt3(\theta-\frac{\pi}{6})}$. Find $Dg(\sqrt3,1)$ (this is Exercise 4.22).

Solution: Note that $(x,y)=(\sqrt3,1)$ when $(r,\theta)=\big(2,\frac{\pi}{6}\big)$ and then, by the Chain Rule,

$$Dh=Dg\cdot Df$$

$$\begin{pmatrix}\frac{\partial h}{\partial r} & \frac{\partial h}{\partial \theta}\end{pmatrix}=\begin{pmatrix}\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{pmatrix}$$

$$\Big(2r\,e^{\sqrt3(\theta-\frac{\pi}{6})},\ \sqrt3r^2e^{\sqrt3(\theta-\frac{\pi}{6})}\Big)=\Big(\frac{\partial g}{\partial x},\frac{\partial g}{\partial y}\Big)\begin{pmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix}$$

$$\big(4\,,\,4\sqrt3\big)=\Big(\frac{\partial g}{\partial x},\frac{\partial g}{\partial y}\Big)\begin{pmatrix}\frac{\sqrt3}{2} & -1\\ \frac12 & \sqrt3\end{pmatrix}$$

and so

$$\nabla g^T=\big(\tfrac{\partial g}{\partial x},\tfrac{\partial g}{\partial y}\big)=\big(4,3\sqrt3\big)\begin{pmatrix}\frac{\sqrt3}{2} & -1\\ \frac12 & \sqrt3\end{pmatrix}^{-1}=\big(4,4\sqrt3\big)\cdot\tfrac12\begin{pmatrix}\sqrt3 & 1\\ -\frac12 & \frac{\sqrt3}{2}\end{pmatrix}=\big(\sqrt3,5\big).$$

(c) Let $f(x,y,z)=x\,\sin(y^2-2xz)$ and let $\alpha(t)=\big(\sqrt t\,,\,\frac12t\,,\,e^{(t-4)/4}\big)$. Find the rate of change of f as we move along the curve $\alpha(t)$ when $t=4$ (this is Exercise 4.25).

Solution: We have $\alpha(4)=(2,2,1)$ and $f(\alpha(4))=f(2,2,1)=2\sin0=0$, and we have

$$\begin{aligned}Df(x,y,z)&=\big(\sin(y^2-2xz)-2xz\cos(y^2-2xz)\ ,\ 2xy\cos(y^2-2xz)\ ,\ -2x^2\cos(y^2-2xz)\big)\\ Df(2,2,1)&=(-4,8,-8)\\ \alpha'(t)&=\big(\tfrac{2}{2\sqrt t},\tfrac12,\tfrac14\,e^{(t-4)/4}\big)^T\\ \alpha'(4)&=\big(\tfrac14,\tfrac12,\tfrac14\big)^T.\end{aligned}$$

For $\beta(t)=f\big(\alpha(t)\big)$ we have $\beta'(t)=Df(\alpha(t))\alpha'(t)$, so the rate of change of f as we move along $\alpha(t)$ is

$$\beta'(4)=Df(2,2,1)\alpha'(4)=\big(-4,8,-8\big)\begin{pmatrix}\frac14\\ \frac12\\ \frac14\end{pmatrix}=1.$$

4: (a) Consider the surface $z = f(x,y)$ where $f(x,y) = \dfrac{4}{2+x^4+x^2+y^2}$. An ant walks counterclockwise around the curve of intersection of the above surface $z = f(x,y)$ with the cylinder $(x-2)^2+y^2 = 5$. Find the value of $\tan\theta$, where θ is the angle (from the horizontal) at which the ant is ascending when it is at the point $\left(1,2,\frac{1}{2}\right)$.

Solution: We have $\frac{\partial f}{\partial x} = \frac{-16x^3-8x}{(2+x^4+x^2+y^2)^2}$ and $\frac{\partial f}{\partial y} = \frac{-8y}{(2+x^4+x^2+y^2)^2}$, and so we have $\frac{\partial f}{\partial x}(1,2) = -\frac{24}{64} = -\frac{3}{8}$ and $\frac{\partial f}{\partial y}(1,2) = -\frac{16}{64} = -\frac{1}{4}$. Thus

$$\nabla f(1,2) = \left(-\tfrac{3}{8}, -\tfrac{1}{4}\right)^T.$$

Looking down from above, the ant moves counterclockwise around the circle $(x-2)^2+y^2 = 5$. The radius vector from the center $(2,0)$ to the point $(1,2)$ is the vector $r = (1,2)-(2,0) = (-1,2)$, and the unit tangent vector perpendicular to r, in the direction in which the ant moves, is $v = \frac{1}{\sqrt{5}}(-2,-1)$. If we parametrize the circle, in the counterclockwise direction, by $\alpha(t) = (x(t),y(t))$ with $\alpha(0) = (1,2)$, then we will have $\frac{\alpha'(0)}{|\alpha'(0)|} = v = \frac{1}{\sqrt{5}}(-2,-1)$. The ant moves along the curve $\gamma(t) = \big(x(t),y(t),z(t)\big) = \big(\alpha(t), f(\alpha(t))\big)$, the tangent vector at $t=0$ is $\gamma'(0) = \big(\alpha'(0), Df(\alpha(0))\alpha'(0)\big)$. The angle of inclination θ of $\gamma'(0)$ is given by

$$\tan\theta = \frac{Df(\alpha(0))\alpha'(0)}{|\alpha'(0)|} = \nabla f(1,2) \bullet \frac{\alpha'(0)}{|\alpha'(0)} = D_v f(1,2) = \left(-\tfrac{3}{8}, -\tfrac{1}{4}\right) \bullet \tfrac{1}{\sqrt{5}}\left(-2,-1\right) = \tfrac{1}{\sqrt{5}}.$$

(b) Consider the surface $z = f(x,y)$ where $f(x,y) = \dfrac{6x}{1+x^2+y^2}$. Show that any circle which passes through the points $(1,0)$ and $(-1,0)$ is a curve of steepest descent, that is for any point (x,y) on any circle C through the two points $(-1,0)$ and $(1,0)$, the slope of C at (x,y) is equal to the slope of the gradient vector $\nabla f(x,y)$.

Solution: The circle through $(1,0)$ and $(-1,0)$ with center at $(0,c)$ has equation $x^2+(y-c)^2 = 1+c^2$. Differentiate this equation (with respect to x) to get $2x+2(y-c)y' = 0$. This shows that at the point (x,y) on the circle, the slope of the circle is $y' = -\frac{x}{y-c}$. On the other hand, $\nabla f(x,y) = \frac{6}{(1+x^2+y^2)^2}(1-x^2+y^2, -2xy)$, so the gradient vector has slope $m = -\frac{2xy}{1-x^2+y^2}$. When (x,y) is on the circle, we have $x^2+(y-c)^2 = 1+c^2$, so $x^2+y^2-2cy = 1$, and so we can put $1-x^2 = y^2-2cy$ into the formula for m to get $m = -\frac{2xy}{y^2-2cy+y^2} = \frac{-x}{y-c}$. Thus the slope of the circle at (x,y) is equal to the slope of the gradient vector at (x,y), as required.