MATH 247 Calculus 3, Solutions to Assignment 2

1: (a) Prove that $\overline{B(a,r)} = \overline{B}(a,r)$ for all $a \in \mathbb{R}^n$ and all $r > 0$.

Solution: Let $a \in \mathbb{R}^n$ and let $r > 0$. Since $\overline{B}(a,r)$ is closed and $B(a,r) \subseteq \overline{B}(a,r)$, we have $\overline{B(a,r)} \subseteq \overline{B}(a,r)$. We need to show that $\overline{B}(a,r) \subseteq \overline{B(a,r)}$. Let $b \in \overline{B}(a,r)$, that is let $b \in \mathbb{R}^n$ with $|b-a| \leq r$. If $|b-a| < r$ then we have $b \in B(a,r)$ and hence $b \in \overline{B(a,r)}$. Suppose that $|b-a| = r$. For each $n \in \mathbb{Z}^+$, let $x_n = \frac{1}{n}\,a + \frac{n-1}{n}\,b$. Note that $x_n - a = \frac{n-1}{n}(b-a)$ and hence $|x_n - a| = \frac{n-1}{n}\,r < r$ so that $x_n \in B(a,r)$. Also note that $x_n - b = \frac{1}{n}(a-b)$ so that $|x_n - b| = \frac{1}{n}\,r$, so that $x_n \neq b$ and $x_n \to b$ in $\mathbb{R}^n$. Since each $x_n \in B(a,r)$ and $x_n \neq b$ and $x_n \to b$, it follows that $b \in B(a,r)'$ (that is, b is a limit point of $B(a,r)$), and hence $b \in \overline{B(a,r)}$. Thus $\overline{B}(a,r) \subseteq \overline{B(a,r)}$, as required.

(b) Determine whether for every subset $P \subseteq \mathbb{R}^n$, we have $\overline{B_P(a,r)} = \overline{B}_P(a,R)$ for all $a \in P$ and all $r > 0$.

Solution: This is not true. For example, when $a, b \in \mathbb{R}^n$ with $a \neq b$, and $P = \{a, b\}$, and $r = |b-a|$, we have $B_P(a,r) = \{a\}$ which is closed (both in P and in $\mathbb{R}^n$) so that $\overline{B_P(a,r)} = \{a\}$, but we have $\overline{B}_P(a,r) = \{a,b\}$.

(c) Let $A \subseteq P \subseteq \mathbb{R}^n$. Prove that A is compact in P if and only if A is compact in $\mathbb{R}^n$.

Solution: Suppose that A is compact in P. Let T be an open cover for A in $\mathbb{R}^n$. For each $V \in T$, let $U_V = V \cap P$. By Theorem 2.31, each set U_V is open in P. Since $A \subseteq P$ and $A \subseteq \bigcup_{V \in T} V$, we also have $A \subseteq \bigcup_{V \in T}(V \cap P) = \bigcup_{V \in T} U_V$. Thus the set $S = \left\{ U_V \,\middle|\, V \in T \right\}$ is an open cover for A in P. Since A is compact in P we can choose a finite subcover, say $\left\{ U_{V_1}, \cdots U_{V_n} \right\}$ of S, where each $V_i \in T$. Since $A \subseteq \bigcup_{i=1}^n U_{V_i} = \bigcup_{i=1}^n (V_i \cap P)$, we also have $A \subseteq \bigcup_{i=1}^n V_i$ and so $\{V_1, \cdots, V_n\}$ is a finite subcover of T.

Suppose, conversely, that A is compact in $\mathbb{R}^n$. Let S be an open cover for A in P. For each $U \in S$, by Theorem 2.31, we can choose an open set V_U in $\mathbb{R}^n$ such that $U = V_U \cap P$. Then $T = \left\{ V_U \,\middle|\, U \in S \right\}$ is an open cover of A in $\mathbb{R}^n$. Since A is compact in $\mathbb{R}^n$ we can choose a finite subcover, say $\left\{ V_{U_1}, \cdots, V_{U_n} \right\}$ of T, where each $U_i \in S$. Then $A \subseteq \bigcup_{i=1}^n (V_{U_i} \cap P) = \bigcup_{i=1}^n U_i$ and so $\{U_1, \cdots, U_n\}$ is a finite subcover of S in P.

2: (a) Let $A \subseteq \mathbb{R}^n$ be compact and let S be an open cover of A. Show that there exists $r > 0$ such that for every $a \in A$ there exists $U \in S$ such that $B(a,r) \subseteq U$.

Solution: For each $p \in A$, since S is an open cover for A we can choose $U_p \in S$ with $p \in U_p$ and then, since U_p is open we can choose $r_p > 0$ so that $B(p, 2r_p) \subseteq U_p$. Note that the set $T = \big\{B(p, r_p)\big| p \in A\big\}$ is an open cover for A. Since A is compact, we can choose a finite subcover, say $\big\{B(p_1, r_{p_1}), \cdots, B(p_\ell, r_{p_\ell})\big\}$ of T for A, with each $p_k \in A$. Let $r = \min\{r_{p_1}, \cdots, r_{p_\ell}\}$. We claim that for every $a \in A$ there exists $U \in S$ such that $B(a,r) \subseteq U$. Let $a \in A$. Choose an index k such that $a \in B(p_k, r_{p_k})$, and let $U = U_{p_k} \in S$. For all $x \in B(a,r)$ we have $|x - p_k| \leq |x - a| + |a - p_k| \leq r + r_{p_k} \leq 2r_{p_k}$ and hence $x \in B(p_k, 2r_{p_k}) \subseteq U_{p_k} = U$. This shows that $B(a,r) \subseteq U$, as required.

(b) Let $C_1, C_2, C_3, \cdots$ be non-empty closed sets in $\mathbb{R}^n$ with $C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots$. Show that if each set C_k is compact then $\bigcap\limits_{k=1}^{\infty} C_k \neq \emptyset$, and find an example where the sets C_k are not compact and we have $\bigcap\limits_{k=1}^{\infty} C_k = \emptyset$.

Solution: Suppose that each set C_k is compact, and suppose, for a contradiction, that $\bigcap\limits_{k=1}^{\infty} C_k = \emptyset$. Then

$$\mathbb{R}^n = \emptyset^{\,c} = \Big(\bigcap_{k=1}^{\infty} C_k\Big)^c = \bigcup_{k=1}^{\infty} {C_k}^c = {C_1}^c \cup \bigcup_{k=2}^{\infty} {C_k}^c.$$

It follows that $C_1 \subseteq \bigcup\limits_{k=2}^{\infty} {C_k}^c$ since given $a \in C_1$ we have $a \in {C_1}^c \cup \bigcup\limits_{k=2}^{\infty} {C_k}^c$ but $a \notin {C_1}^c$, and so $a \in \bigcup\limits_{k=2}^{\infty} {C_k}^c$. Thus $S = \big\{{C_2}^c, {C_3}^c, {C_4}^c, \cdots\big\}$ is an open cover for C_1. Since C_1 is compact, we can choose a finite sub-cover $T = \big\{{C_{k_1}}^c, {C_{k_2}}^c, \cdots, {C_{k_\ell}}^c\big\}$ say with $2 \leq k_1 < k_2 < \cdots < k_\ell$. Since T covers C_1 we have $C_1 \subseteq \bigcup\limits_{i=1}^{\ell} {C_{k_i}}^c$. Since $C_{k_1} \supseteq C_{k_2} \supset \cdots \supset C_{k_\ell}$ we have ${C_{k_1}}^c \subseteq {C_{k_2}}^c \subseteq \cdots \subseteq {C_{k_\ell}}^c$ and hence $\bigcup\limits_{i=1}^{\ell} {C_{k_i}}^c = {C_{k_\ell}}^c$. Thus we obtain $C_1 \subseteq {C_{k_\ell}}^c$, or equivalently $C_1 \cap C_{k_\ell} = \emptyset$. But this is not possible since $C_1 \cap C_{k_\ell} = C_{k_\ell} \neq \emptyset$.

Note that the sets $C_k = \mathbb{R}^m \setminus B(0,n)$ are closed in $\mathbb{R}^m$ with $C_1 \supseteq C_2 \supseteq \cdots$, but $\bigcap\limits_{k=1}^{\infty} C_k = \emptyset$.

3: Note that $\mathbb{C}=\mathbb{R}^2$ so a sequence in $\mathbb{C}$ is a sequence in $\mathbb{R}^2$.

(a) For $k\geq 0$, let $x_k=\left(\frac{3+i\sqrt{3}}{4}\right)^k\in\mathbb{C}$, and for $n\geq 0$, let $s_n=\sum\limits_{k=0}^{n}x_k\in\mathbb{C}$. Use the definition of the limit (for a sequence in $\mathbb{R}^2$) to find $a,b\in\mathbb{R}$ such that $\lim\limits_{n\to\infty}s_n=a+ib$.

Solution: From the formula for the sum of a geometric series, or by noting that $s_n=\sum\limits_{k=0}^{n}\left(\frac{3+i\sqrt{3}}{4}\right)^k$ and $\left(\frac{3+i\sqrt{3}}{4}\right)s_n=\sum\limits_{k=0}^{n}\left(\frac{3+i\sqrt{4}}{4}\right)^{k+1}$, so that $s_n-\left(\frac{3+i\sqrt{3}}{4}\right)s_n=1-\left(\frac{3+i\sqrt{3}}{4}\right)^{n+1}$, we have

$$s_n=\frac{1-\left(\frac{3+i\sqrt{3}}{4}\right)^{n+1}}{1-\frac{3+i\sqrt{3}}{4}}=\frac{1-\left(\frac{3+i\sqrt{3}}{4}\right)^{n+1}}{\frac{1-i\sqrt{3}}{4}}\cdot\frac{1+i\sqrt{3}}{1+i\sqrt{3}}=(1+i\sqrt{3})\Big(1-\left(\frac{3+i\sqrt{3}}{4}\right)^{n+1}\Big)$$

and hence

$$\left|s_n-(1+i\sqrt{3})\right|=|1+i\sqrt{3}|\left|\frac{3+i\sqrt{3}}{4}\right|^{n+1}=2\cdot\left(\frac{\sqrt{3}}{2}\right)^{n+1}.$$

It follows that $\lim\limits_{n\to\infty}s_n=1+i\sqrt{3}$: indeed given $\epsilon>0$ we can choose $m\in\mathbb{N}$ so that $\left(\frac{\sqrt{3}}{2}\right)^m<\frac{\epsilon}{2}$, and then when $n\geq m$ we have $\left|s_n-(1+i\sqrt{3})\right|=2\cdot\left(\frac{\sqrt{3}}{2}\right)^n\leq 2\cdot\left(\frac{\sqrt{3}}{2}\right)^m<\epsilon$.

(b) Let $c=\frac{2-i}{8}\in\mathbb{C}$. Let $(z_n)_{n\geq 0}$ be the sequence in $\mathbb{C}$ given by $z_0=0$ and $z_{n+1}={z_n}^2+c$ for $n\geq 0$. Determine whether $(z_n)_{n\geq 0}$ converges in $\mathbb{C}$ and, if so, find $\lim\limits_{n\to\infty}z_n$ in $\mathbb{C}$.

Solution: If (z_n) converges with $z_n\to w$ in $\mathbb{C}$, then taking the limit on each side of the equality $z_{n+1}={z_n}^2+c$ gives $w=w^2+c$. By the Quadratic Formula, we have $w=w^2+c\iff w^2-w+c=0\iff w=\frac{1\pm\sqrt{1-4c}}{2}$, (where $\sqrt{1-4c}$ is one of the two square roots of $1-4c$ in $\mathbb{C}$). Note that $1-4c=1-\frac{2-i}{2}=\frac{i}{2}=\left(\frac{1+i}{2}\right)^2$, so we must have $w=\frac{1\pm\frac{1+i}{2}}{2}=\frac{2\pm(1+i)}{4}$, that is $w=\frac{3+i}{4}$ or $w=\frac{1-i}{4}$.

Let $w=\frac{1-i}{4}$. We claim that $z_n\to w$. Note that $z_0-w=0-w=\frac{-1+i}{4}$ so that $|z_0-w|=\frac{1}{2\sqrt{2}}$ and $z_1-w=c-w=\frac{2-i}{8}-\frac{1-i}{4}=\frac{i}{8}$ so that $|z_1-w|=\frac{1}{8}$. Let $n\geq 1$ and suppose, inductively, that $|z_n-w|\leq\frac{1}{8}$ and that $|z_n-w|\leq\frac{1}{8}\left(\frac{5}{4\sqrt{2}}\right)^{n-1}$. We have

$$z_{n+1}-w={z_n}^2+c-w={z_n}^2+\frac{i}{8}={z_n}^2-w^2=(z_n-w)(z_n+w)=(z_n-w)\big((z_n-w)+2w\big)$$

so that

$$|z_{n+1}-w|\leq|z_n-w|\big(|z_n-w|+|2w|\big)=|z_n-w|\big(|z_n-w|+\tfrac{1}{\sqrt{2}}\big).$$

Using the first induction hypotheses gives

$$|z_{n+1}-w|\leq|z_n-w|\big(\tfrac{1}{8}+\tfrac{1}{\sqrt{2}}\big)\leq|z_n-w|\big(\tfrac{1}{4\sqrt{2}}+\tfrac{1}{\sqrt{2}}\big)=\tfrac{5}{4\sqrt{2}}\,|z_n-w|.$$

Using this with the first induction hypothesis again gives $|z_{n+1}-w|\leq\frac{5}{4\sqrt{2}}|z_n-w|\leq|z_n-w|\leq\frac{1}{8}$, and using it with the second induction hypothesis gives $|z_{n+1}-w|\leq\frac{5}{4\sqrt{2}}\,|z_n-w|\leq\frac{5}{4\sqrt{2}}\cdot\frac{1}{8}\left(\frac{5}{\sqrt{2}}\right)^{n-1}=\frac{1}{8}\left(\frac{5}{4\sqrt{2}}\right)^n$. Thus, by induction, we have $|z_n-w|\leq\frac{1}{8}\left(\frac{5}{4\sqrt{2}}\right)^{n-1}$ for all $n\geq 1$.

It follows that $z_n\to w$, as claimed: indeed given $\epsilon>0$, since $\frac{5}{4\sqrt{2}}<1$ so that $\left(\frac{5}{4\sqrt{2}}\right)^{n-1}\to 0$, we can choose $m\in\mathbb{Z}^+$ so that $\left(\frac{5}{4\sqrt{2}}\right)^{m-1}<8\,\epsilon$ and then for $n\geq m$ we have

$$|z_n-w|\leq\tfrac{1}{8}\left(\tfrac{5}{4\sqrt{2}}\right)^{n-1}\leq\tfrac{1}{8}\left(\tfrac{5}{4\sqrt{2}}\right)^{m-1}<\epsilon.$$

4: Let $\mathbb{R}^\omega$ be the set of all sequences in $\mathbb{R}$, that is $\mathbb{R}^\omega = \left\{x = (x_j)_{j\geq 1} \,\middle|\, \text{each } x_j \in \mathbb{R}\right\}$ and let $\mathbb{R}^\infty$ be the set of eventually zero sequences in $\mathbb{R}$, that is $\mathbb{R}^\infty = \left\{x = (x_j)_{j\geq 1} \in \mathbb{R}^\omega \,\middle|\, \exists\, m \in \mathbb{Z}^+ \;\forall\, j \in \mathbb{Z}^+ \;(j \geq m \Longrightarrow x_j = 0)\right\}$. For $x, y \in \mathbb{R}^\infty$, define $x \bullet y = \sum_{n=1}^\infty x_n y_n$ and $|x| = \left(x \bullet x\right)^{1/2}$.

When $(x_n)_{n\geq 1}$ is a sequence in $\mathbb{R}^\infty$, each $x_n \in \mathbb{R}^\infty$, and we can write $x_n = (x_{n,j})_{j\geq 1} = (x_{n,1}, x_{n,2}, x_{n,3}, \cdots)$. For a sequence $(x_n)_{n\geq 1}$ in $\mathbb{R}^\infty$ and an element $a \in \mathbb{R}^\infty$, we say the sequence $(x_n)_{n\geq 1}$ *converges* to a in $\mathbb{R}^\infty$, and we write $x_n \to a$ in $\mathbb{R}^\infty$ or $\lim\limits_{n\to\infty} x_n = a$ in $\mathbb{R}^\infty$, when $\forall\, \epsilon > 0 \;\exists\, m \in \mathbb{Z}^+ \;\forall\, n \in \mathbb{Z}^+ \;(n \geq m \Longrightarrow |x_n - a| < \epsilon)$, we say that $(x_n)_{n\geq 1}$ is *bounded* when $\exists\, r \geq 0 \;\forall\, n \in \mathbb{Z}^+ \;|x_n| \leq r$, and we say that $(x_n)_{n\geq 1}$ is *Cauchy* when $\forall\, \epsilon > 0 \;\exists\, m \in Z^+ \;\forall\, k, \ell \in \mathbb{Z}^+ \;(k, \ell \geq m \Longrightarrow |x_k - x_\ell| < \epsilon)$.

(a) Prove that for all sequences $(x_n)_{n\geq 1}$ in $\mathbb{R}^\infty$, and all $a \in \mathbb{R}^\infty$, if $\lim\limits_{n\to\infty} x_n = a$ in $\mathbb{R}^\infty$ then $\lim\limits_{n\to\infty} x_{n,j} = a_j$ for all $j \in \mathbb{Z}^+$, but that the converse does not hold.

Solution: Let $(x_n)_{n\geq 1}$ be a sequence in $\mathbb{R}^\infty$ and let $a \in \mathbb{R}^\infty$. Suppose that $\lim\limits_{n\to\infty} x_n = a$ in $\mathbb{R}^\infty$. We claim that $\lim\limits_{n\to\infty} x_{n,j} = a_j$ for all $j \in \mathbb{Z}^+$. Let $j \in \mathbb{Z}^+$. Note that $|x_{n,j} - a_j|^2 \leq \sum_{i=1}^\infty (x_{n,i} - a_i)^2 = |x_n - a|^2$. Since $|x_{n,j} - a_j| \leq |x_n - a|$ and $\lim\limits_{n\to\infty} x_n = a$ in $\mathbb{R}^\infty$, it follows that $\lim\limits_{n\to\infty} x_{n,j} = a_k$ in $\mathbb{R}$: indeed given $\epsilon > 0$, we can choose $m \in \mathbb{Z}^+$ so that $n \geq m \Longrightarrow |x_n - a| < \epsilon$, and then, for $n \geq m$, we have $|x_{n,j} - a_j| \leq |x_n - a| < \epsilon$.

To see that the converse does not hold, for each $n \in \mathbb{Z}^+$, let $x_n = \frac{1}{\sqrt{n}} \sum_{k=1}^n e_k = \left(\frac{1}{\sqrt{n}}, \cdots, \frac{1}{\sqrt{n}}, 0, 0, \cdots\right)$, where $e_k = (0, 0, \cdots, 0, 1, 0, 0, \cdots)$ is the k^{th} standard basis vector for $\mathbb{R}^\infty$. For each index $j \in \mathbb{Z}^+$ we have $x_{n,j} = \frac{1}{\sqrt{n}}$ for all $n \geq j$ so that $\lim\limits_{n\to\infty} x_{n,j} = 0$ in $\mathbb{R}$. But for $a = 0 = (0, 0, 0, \cdots)$ we do not have $\lim\limits_{n\to\infty} x_n = a$ in $\mathbb{R}^\infty$ because $|x_n - 0| = |x_n| = 1$ for all $n \in \mathbb{Z}^+$.

(b) Prove that for all sequences $(x_n)_{n\geq 1}$ in $\mathbb{R}^\infty$, if the sequence $(x_n)_{n\geq 1}$ converges in $\mathbb{R}^\infty$ (to some $a \in \mathbb{R}^\infty$) then it is Cauchy, but that the converse does not hold.

Solution: Let $(x_n)_{n\geq 1}$ be a sequence in $\mathbb{R}^\infty$. Suppose that $(x_n)_{n\geq 1}$ converges in $\mathbb{R}^\infty$ and let $a = \lim\limits_{n\to\infty} x_n$ in $\mathbb{R}^\infty$. Let $\epsilon > 0$. Choose $m \in \mathbb{Z}^+$ so that $n \geq m \Longrightarrow |x_n - a| < \frac{\epsilon}{2}$. Then when $k, \ell \geq m$ we have $|x_k - x_\ell| = \left|(x_k - a) - (x_\ell - a)\right| \leq |x_k - a| + |x_\ell - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Thus $(x_n)_{n\geq 1}$ is Cauchy.

To see that the converse does not hold, for each $n \in \mathbb{Z}^+$ let $x_n = \sum_{k=1}^n \frac{1}{2^k} e_k = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots, \frac{1}{2^n}, 0, 0, \cdots\right)$. We claim that $(x_n)_{n\geq 1}$ is Cauchy. Let $\epsilon > 0$. Choose $m \in \mathbb{Z}^+$ so that $\frac{1}{2^m} < \epsilon$. Let $k, \ell \in \mathbb{Z}^+$ with $m \leq k < \ell$. Then we have $|x_k - x_\ell|^2 = \left|\sum_{j=k+1}^\ell \frac{1}{2^j} e_j\right|^2 = \sum_{j=k+1}^\ell \frac{1}{4^j} \leq \sum_{j=k+1}^\infty \frac{1}{4^j} = \frac{1}{4^k}$ so that $|x_k - x_\ell| \leq \frac{1}{2^k} \leq \frac{1}{2^m} < \epsilon$. Thus $(x_n)_{n\geq 1}$ is Cauchy, as claimed. Suppose, for a contradiction, that $(x_n)_{n\geq 1}$ converges in $\mathbb{R}^\infty$ and let $a = \lim\limits_{n\to\infty} x_n \in \mathbb{R}^\infty$. Note that for each $j \in \mathbb{Z}^+$, we have $x_{n,j} = \frac{1}{2^j}$ for all $n \geq j$ so that $\lim\limits_{n\to\infty} x_{n,j} = \frac{1}{2^j}$. By Part (a), for each $j \in \mathbb{Z}^+$ we must have $a_j = \lim\limits_{n\to\infty} x_{n,j} = \frac{1}{2^j}$ so that $a = \sum_{j=1}^\infty \frac{1}{2^j} e_j = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots\right)$. But then $a \notin \mathbb{R}^\infty$, which gives the desired contradiction.

(c) Determine whether every bounded sequence $(x_n)_{n\geq 1}$ in $\mathbb{R}^\infty$ has a convergent subsequence $(x_{n_k})_{k\geq 1}$ $\mathbb{R}^\infty$.

Solution: This is not true. For example, consider the sequence $x_n = e_n = (0, \cdots, 0, 1, 0, \cdots)$ for $n \in \mathbb{Z}^+$. Note that $(x_n)_{n\geq 1}$ is bounded since $|x_n| = 1$ for all $n \in \mathbb{Z}^+$. Let $(x_{n_k})_{k\geq 1}$ be any subsequence. Note that for $k, \ell \in \mathbb{Z}^+$ with $k \neq \ell$ we have $|x_{n_k} - x_{n_\ell}| = |e_{n_k} - e_{n_\ell}| = \sqrt{2}$, and so the sequence $(x_{n_k})_{k\geq 1}$ is not Cauchy (if it was Cauchy, then we would be able to choose $k, \ell \in \mathbb{Z}^+$ with $k < \ell$ such that $|x_{n_k} - x_{n_\ell}| < \sqrt{2}$). Since $(x_{n_k})_{k\geq 1}$ is not Cauchy, it does not converge, by Part (b).