MATH 247 Calculus 3, Assignment 2 Due: Fri Feb 16, 11:00 pm.

1: (a) Prove that $\overline{B(a,r)} = \overline{B}(a,r)$ for all $a \in \mathbb{R}^n$ and all $r > 0$.

(b) Determine whether for every subset $P \subseteq \mathbb{R}^n$, we have $\overline{B_P(a,r)} = \overline{B}_P(a,r)$ for all $a \in P$ and all $r > 0$.

(c) Let $A \subseteq P \subseteq \mathbb{R}^n$. Prove that A is compact in P if and only if A is compact in $\mathbb{R}^n$ (this is Theorem 2.32).

2: (a) Let $A \subseteq \mathbb{R}^n$ be compact and let S be an open cover of A. Show that there exists $r > 0$ such that for every $a \in A$ there exists $U \in S$ such that $B(a,r) \subseteq U$.

(b) Let $C_1, C_2, C_3, \cdots$ be non-empty closed sets in $\mathbb{R}^n$ with $C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots$. Show that if each set C_k is compact then $\bigcap\limits_{k=1}^{\infty} C_k \neq \emptyset$, and find an example where the sets C_k are not compact and we have $\bigcap\limits_{k=1}^{\infty} C_k = \emptyset$.

3: Note that $\mathbb{C} = \mathbb{R}^2$ so a sequence in $\mathbb{C}$ is a sequence in $\mathbb{R}^2$.

(a) For $k \geq 0$, let $x_k = \left(\frac{3+i\sqrt{3}}{4}\right)^k \in \mathbb{C}$, and for $n \geq 0$, let $s_n = \sum\limits_{k=0}^{n} x_k \in \mathbb{C}$. Use the definition of the limit (for a sequence in $\mathbb{R}^2$) to find $a, b \in \mathbb{R}$ such that $\lim\limits_{n\to\infty} s_n = a + ib$.

(b) Let $c = \frac{2-i}{8} \in \mathbb{C}$. Let $(z_n)_{n\geq 0}$ be the sequence in $\mathbb{C}$ given by $z_0 = 0$ and $z_{n+1} = {z_n}^2 + c$ for $n \geq 0$. Determine whether $(z_n)_{n\geq 0}$ converges in $\mathbb{C}$ and, if so, find $\lim\limits_{n\to\infty} z_n$ in $\mathbb{C}$.

4: Let $\mathbb{R}^\omega$ be the set of all sequences in $\mathbb{R}$, that is $\mathbb{R}^\omega = \{x = (x_j)_{j\geq 1} \,|\, \text{each } x_j \in \mathbb{R}\}$ and let $\mathbb{R}^\infty$ be the set of eventually zero sequences in $\mathbb{R}$, that is $\mathbb{R}^\infty = \{x = (x_j)_{j\geq 1} \in \mathbb{R}^\omega \,|\, \exists\, m \in \mathbb{Z}^+ \ \forall\, j \in \mathbb{Z}^+ \ (j \geq m \Longrightarrow x_j = 0)\}$. For $x, y \in \mathbb{R}^\infty$, define $x \bullet y = \sum_{n=1}^{\infty} x_n y_n$ and $|x| = (x \bullet x)^{1/2}$.

When $(x_n)_{n\geq 1}$ is a sequence in $\mathbb{R}^\infty$, each $x_n \in \mathbb{R}^\infty$, and we can write $x_n = (x_{n,j})_{j\geq 1} = (x_{n,1}, x_{n,2}, x_{n,3}, \cdots)$. For a sequence $(x_n)_{n\geq 1}$ in $\mathbb{R}^\infty$ and an element $a \in \mathbb{R}^\infty$, we say the sequence $(x_n)_{n\geq 1}$ *converges* to a in $\mathbb{R}^\infty$, and we write $x_n \to a$ in $\mathbb{R}^\infty$ or $\lim\limits_{n\to\infty} x_n = a$ in $\mathbb{R}^\infty$, when $\forall\, \epsilon > 0 \ \exists\, m \in \mathbb{Z}^+ \ \forall\, n \in \mathbb{Z}^+ \ (n \geq m \Longrightarrow |x_n - a| < \epsilon)$, we say that $(x_n)_{n\geq 1}$ is *bounded* when $\exists\, r \geq 0 \ \forall\, n \in \mathbb{Z}^+ \ |x_n| \leq r$, and we say that $(x_n)_{n\geq 1}$ is *Cauchy* when $\forall\, \epsilon > 0 \ \exists\, m \in Z^+ \ \forall\, k, \ell \in \mathbb{Z}^+ \ (k, \ell \geq m \Longrightarrow |x_k - x_\ell| < \epsilon)$.

(a) Prove that for all sequences $(x_n)_{n\geq 1}$ in $\mathbb{R}^\infty$, and all $a \in \mathbb{R}^\infty$, if $\lim\limits_{n\to\infty} x_n = a$ in $\mathbb{R}^\infty$ then $\lim\limits_{n\to\infty} x_{n,j} = a_j$ for all $j \in \mathbb{Z}^+$, but that the converse does not hold.

(b) Prove that for all sequences $(x_n)_{n\geq 1}$ in $\mathbb{R}^\infty$, if the sequence $(x_n)_{n\geq 1}$ converges in $\mathbb{R}^\infty$ (to some $a \in \mathbb{R}^\infty$) then it is Cauchy, but that the converse does not hold.

(c) Determine whether every bounded sequence $(x_n)_{n\geq 1}$ in $\mathbb{R}^\infty$ has a convergent subsequence $(x_{n_k})_{k\geq 1}$ $\mathbb{R}^\infty$.