MATH 247 Calculus 3, Solutions to Assignment 2.5

1: (a) Let $f(x,y)=\dfrac{xy^2}{x^2+2y^2}$ for $(x,y)\neq(0,0)$. Determine whether $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists and, if so, find it.

Solution: We claim that $\lim\limits_{(x,y)\to(0,0)} f(x,y)=0$. For all x,y we have $|x|=\sqrt{x^2}\leq\sqrt{x^2+y^2}$ and $y^2\leq x^2+y^2$ and $x^2+2y^2\geq x^2+y^2$ and so

$$\big|f(x,y)-0\big|=\left|\frac{xy^2}{x^2+2y^2}\right|=\frac{|x|y^2}{x^2+2y^2}\leq\frac{\sqrt{x^2+y^2}\,(x^2+y^2)}{x^2+y^2}=\sqrt{x^2+y^2}\,.$$

Thus given $\epsilon>0$ we can choose $\delta=\epsilon$ and then for all x with $0<\big|(x,y)-(0,0)\big|<\delta$ we have

$$\big|f(x,y)-0\big|\leq\sqrt{x^2+y^2}<\delta=\epsilon\,.$$

(b) Let $f(x,y)=\dfrac{x\sqrt{y}}{x^2+y}$ for $y>0$. Determine whether $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists and, if so, find it.

Solution: Suppose, for a contradiction, that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists. Let $\alpha(t)=(t,0)$ for $t>0$. Since $\alpha(t)\neq(0,0)$ for $t>0$ and $\lim\limits_{t\to0}\alpha(t)=(0,0)$ it follows, by Part 1 of Theorem 3.31 (Composition and Limits), that $\lim\limits_{(x,y)\to(0,0)} f(x,y)=\lim\limits_{t\to0} f(\alpha(t))=\lim\limits_{t\to0} 0=0$. Let $\beta(t)=(t,t^2)$ for $t>0$. Since $\beta(t)\neq(0,0)$ for $t>0$ and $\lim\limits_{t\to0}\beta(t)=(0,0)$, it follows, again by Theorem 3.31, that $\lim\limits_{(x,y)\to(0,0)} f(x,y)=\lim\limits_{t\to0} f(\beta(t))=\lim\limits_{t\to0}=\lim\limits_{t\to0}\frac{1}{2}=\frac{1}{2}$. By Theorem 3.18 (the uniqueness of limits), we cannot have $\lim\limits_{(x,y)\to(0,0)} f(x,y)=0$ and $\lim\limits_{(x,y)\to(0,0)}=\frac{1}{2}$, so we obtain the desired contradiction. Thus $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

(c) Define $f:\mathbb{R}^2\to\mathbb{R}$ by $f(x,y)=\left\{\begin{array}{cl}\dfrac{xy}{x^2-y^2} & \text{if } y\neq\pm x\\ 0 & \text{if } y=\pm x\end{array}\right\}$. Determine where $f(x,y)$ is continuous, that is find all points $(a,b)\in\mathbb{R}^2$ such that f is continuous at (a,b).

Solution: Note that f is continuous for all points (x,y) with $y\neq\pm x$ because elementary functions are continuous in their domains. We claim that f not continuous at any other points.

First, let us show that f is not continuous at $(0,0)$. Suppose, for a contradiction, that f is continuous at $(0,0)$. Define $\alpha:\mathbb{R}\to\mathbb{R}^2$ by $\alpha(t)=(2t,t)$. Since α is continuous (it is elementary) with $\alpha(0)=(0,0)$, it follows, by Part 1 of Corollary 3.32 (Composition of Continuous Functions), that $g=f\circ\alpha$ is continuous at 0. This implies that $g(0)=\lim\limits_{t\to0} g(t)=\lim\limits_{t\to0} f(\alpha(t))=\lim\limits_{t\to0}\frac{2t^2}{4t^2-t^2}=\frac{2}{3}$, but in fact $g(0)=f(\alpha(0))=f(0,0)=0$, which gives the desired contradiction.

Finally, let us show that f is not continuous at any point $(a,\pm a)$ with $a\neq0$. Let $0\neq a\in\mathbb{R}$. Suppose, for a contradiction, that f is continuous at (a,a). Define $\beta:\mathbb{R}\to\mathbb{R}^2$ by $\beta(t)=(a,a)+t(a,-a)$ and note that β is continuous with $\beta(0)=(a,a)$. By Corollary 3.32, the composite $h=f\circ\beta$ is continuous at 0. This implies that $h(0)=\lim\limits_{t\to0} f(\beta(t))=\lim\limits_{t\to0}\frac{(a+ta)(a-ta)}{(a+ta)^2-(a-ta)^2}=\lim\limits_{t\to0}\frac{1-t^2}{4t}$, but this is not possible since $\lim\limits_{t\to0}\frac{1-t^2}{4t}$ does not exist. Similarly, f is not continuous at $(a,-a)$ since, if it was, then for $\gamma(t)=(a,-a)+t(a,a)$ and $k=f\circ\gamma$, we would have $k(0)=\lim\limits_{t\to0} f(\gamma(t))=\lim\limits_{t\to0}\frac{(a+ta)(-a+ta)}{(a+ta)^2-(-a+ta)^2}=\lim\limits_{t\to0}\frac{t^2-1}{4t}$, which does not exist.

2: For each of the following subsets $A \subseteq \mathbb{R}^n$, determine whether A is closed, whether A is compact, and whether A is connected.

(a) $A = \left\{ (a,b,c,d) \in \mathbb{R}^4 \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$.

Solution: We claim that A is closed. For $a,b,c,d \in \mathbb{R}$ we have $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix}$ so that $(a,b,c,d) \in A \iff \big(a^2+bc, ab+bd, ac+cd, bc+d^2\big) = (1,0,0,1)$, and so $A = f^{-1}(p)$ where $f : \mathbb{R}^4 \to \mathbb{R}^4$ is given by $f(a,b,c,d) = \big(a^2+bc, ab+bd, ac+cd, bc+d^2\big)$ and $p = (1,0,0,1) \in \mathbb{R}^4$. The map f is continuous (it is a polynomial map) and $\{p\}$ is closed in $\mathbb{R}^4$, and so $A = f^{-1}\big(\{p\}\big)$ is closed in $\mathbb{R}^4$ (by Theorem 3.36).

Note that A is not bounded because for $r > 0$ we have $(1,r,0,-1) \in A$ and $\big|(1,r,0,-1)\big| = \sqrt{2+r^2} \to \infty$ as $r \to \infty$. Since A is not bounded in $\mathbb{R}^4$, it is not compact (by the Heine Borel Theorem).

We claim that A is not connected. For $a,b,c,d \in \mathbb{R}$, if $(a,b,c,d) \in A$ then we have $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = I$ so that $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm 1$, that is $ad - bc = \pm 1$. It follows that A can be separated in $\mathbb{R}^4$ by the two open sets $U = \big\{(a,b,c,d) \big| ad-bc > 0\big\}$ and $V = \big\{(a,b,c,d) \big| ad-bc < 0\big\}$. Note that U is open because $U = g^{-1}\big((0,\infty)\big)$ where $g : \mathbb{R}^4 \to \mathbb{R}$ is given by $g(a,b,c,d) = ad - bc$ (and g is continuous and $(0,\infty)$ is open), and similarly V is open because $V = g^{-1}\big((-\infty,0)\big)$. And we have $U \cap A \neq \emptyset$ because for example $(1,0,0,1) \in U \cap A$, and we have $V \cap A \neq \emptyset$ because for example $(0,1,1,0) \in V \cap A$. And $A \subseteq U \cup V$ since $(a,b,c,d) \in A \Longrightarrow ad - bc \neq 0$.

(b) A is the set of points $(a,b,c) \in \mathbb{R}^3$ such that the polynomial $p(x) = x^3 + ax^2 + bx + c$ has three distinct real roots which all lie in the closed interval $[-1,1]$.

Solution: We claim that A is not closed in $\mathbb{R}^3$. For $n \in \mathbb{Z}^+$, let $u_n = (a_n, b_n, c_n) = \big(0, -\frac{1}{n^2}, 0\big) \in \mathbb{R}^3$. Note that $u_n \in A$ since the polynomial $p_n(x) = x^3 + a_n x^2 + b_n x + c_n = x^3 - \frac{1}{n^2}x = \big(x + \frac{1}{n}\big)\big(x - 0\big)\big(x - \frac{1}{n}\big)$ has 3 distinct real roots, namely $-\frac{1}{n}$, 0, and $\frac{1}{n}$, which all lie in $[-1,1]$. Note that $\lim\limits_{n\to\infty} u_n = (0,0,0)$ so that $(0,0,0) \in \overline{A}$. But $(0,0,0) \notin A$ because the polynomial $p(x) = x^3 + 0x^2 + 0x + 0 = x^3$ does not have three distinct real roots (it has the single triple root, 0). Thus A is not closed in $\mathbb{R}^3$ (by Theorem 3.11), as claimed. Since A is not closed in $\mathbb{R}^3$, it is not compact (by the Heine-Borel Theorem).

We claim that A is connected. Let $C = \big\{(r,s,t) \in \mathbb{R}^3 \big| -1 \leq r < s < t \leq 1\big\}$ and define $f : C \to A$ by $f(r,s,t) = \big(-(r+s+t), st+tr+rs, -rst\big)$. Note that f is continuous (all polynomial functions are continuous), and f takes values in A and is surjective because $x^3 - (r+s+t)x^2 + (st+tr+rs)x - rst = (x-r)(x-s)(x-t)$. Note that $C = C_1 \cap C_2 \cap C_3 \cap C_4$ where $C_1 = \big\{(r,s,t) \big| -1 \leq r\big\}$, $C_2 = \big\{(r,s,t) \big| r < s\big\}$, $C_3 = \big\{(r,s,t) \big| s < t\big\}$ and $C_4 = \big\{(r,s,t) \big| t \leq 1\big\}$. Each of these sets C_k is easily seen to be convex: for example, C_2 is convex because if $u_1 = (r_1, s_1, t_1) \in C_2$ (so $r_1 < s_2$) and $u_2 = (r_2, s_2, t_2) \in C_2$ (so $r_2 < s_2$) then for all $\lambda \in [0,1]$ we have $(1-\lambda)r_1 + \lambda r_2 < (1-\lambda)s_1 + \lambda s_2$ so that

$$(1-\lambda)u_1 + \lambda u_2 = \big((1-\lambda)r_1 + \lambda r_2, (1-\lambda)s_1 + \lambda s_2, (1-\lambda)t_1 + \lambda t_2\big) \in C_2.$$

Since C is the intersection of four convex sets, it follows that C is convex: indeed given $a, b \in C$, we have $a, b \in C_k$ so that $[a,b] \subseteq C_k$ for every index k, and hence $[a,b] \subseteq C = \bigcap_{k=1}^4 C_k$. Since C is convex, it is path connected, and hence connected. Since f is continuous and C is connected and $A = f(C)$, it follows that A is connected by Part 1 of Theorem 3.37.

3: (a) When $A \subseteq \mathbb{R}^\ell$ is unbounded, $f : A \subseteq \mathbb{R}^\ell \to \mathbb{R}^m$, and $b \in \mathbb{R}^m$, we write $\lim\limits_{x\to\infty} f(x) = b$ when

$$\forall \epsilon > 0\ \exists r > 0\ \ \forall x \in A\ \ \big(|x| \geq r \Longrightarrow |f(x) - b| < \epsilon\big).$$

Show that if $A \subseteq \mathbb{R}^\ell$ is closed and unbounded, and $f : A \subseteq \mathbb{R}^\ell \to \mathbb{R}^m$ is continuous, and $\lim\limits_{x\to\infty} f(x) = b \in \mathbb{R}^m$, then f is uniformly continuous on A.

Solution: Suppose A is closed and unbounded in $\mathbb{R}^\ell$, and $f : A \subseteq \mathbb{R}^\ell \to \mathbb{R}^m$ is continuous with $\lim\limits_{x\to\infty} f(x) = b$. Let $\epsilon > 0$. Since $\lim\limits_{x\to\infty} f(x) = b$ we can choose $r > 0$ such that for all $x \in A$ with $x \geq r$ we have $|f(x) - b| < \frac{\epsilon}{2}$. Since A is closed, the set $B = \overline{B}(0, 3r) \cap A$ is closed, and since B is also bounded, it is compact. Since f is continuous on B, which is compact, it follows that f is uniformly continuous on B, so we can choose $\delta > 0$ with $\delta < r$ such that for all $a, x \in A$, if $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$. Let $a, x \in A$ with $|x - a| < \delta$. If $|a| \leq 2r$ then since $|x - a| < r$ we have $|x| \leq |x - a| + |a| < r + 2r = 3r$, so that $x, a \in B$ with $|x - a| < \delta$, and hence $|f(x) - f(a)| < \epsilon$. If $|a| \geq 2r$ then since $|x - a| < r$ we have $|a| \leq |a - x| + |x|$ so that $|x| \geq |a| - |x - a| > 2r - r = r$, so we have $|a| > r$ and $|x| > r$, and hence $|f(a) - b| < \frac{\epsilon}{2}$ and $|f(x) - b| < \frac{\epsilon}{2}$ so that $|f(x) - f(a)| \leq |f(x) - b| + |b - f(a)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

(b) Show that if $f : A \subseteq \mathbb{R}^\ell \to \mathbb{R}^m$ is uniformly continuous on A then there exists a unique continuous function $g : \overline{A} \subseteq \mathbb{R}^\ell \to \mathbb{R}^m$ with $g(x) = f(x)$ for all $x \in A$, and that g is uniformly continuous on $\overline{A}$.

Solution: Suppose that f is uniformly continuous on A. Note that if $a \in \overline{A}$ then there exists a sequence (x_n) in A such that $x_n \to a$: indeed if $a \in A$ then we can use the constant sequence $x_n = a$ for all n, and if $a \in A'$ then we can choose a sequence in $A \setminus \{a\}$ by Theorem 3.10 (the sequential characterization of limit points).

We claim that when $a \in \overline{A}$ and (x_n) is a sequence in A with $x_n \to a$, the sequence $(f(x_n))$ converges in $\mathbb{R}^m$. Let $\epsilon > 0$. Since f is uniformly continuous on A, we can choose $\delta > 0$ such that for all $x, y \in A$ we have $|x - y| < \delta \Longrightarrow \big|f(x) - f(y)\big| < \epsilon$. Since $x_n \to a$, we can choose $n \in \mathbb{Z}^+$ such that $k \geq n \Longrightarrow |x_n - a| < \frac{\delta}{2}$. Then for $k, \ell \geq n$ we have $|x_k - x_\ell| \leq |x_k - a| + |a - x_\ell| < \frac{\delta}{2} + \frac{\delta}{2} = \delta$, and hence $\big|f(x_k) - f(x_\ell)\big| < \epsilon$. This shows that the sequence $(f(x_n))$ is Cauchy in $\mathbb{R}^m$, and so it converges, as claimed.

We claim that when $a \in \overline{A}$ and (x_n) and (y_n) are two sequences in A with $x_n \to a$ and $y_n \to a$, we have $\lim\limits_{n\to\infty} f(x_n) = \lim\limits_{n\to\infty} f(y_n)$. By the previous paragraph, we know that the sequences $(f(x_n))$ and $f(y_n))$ both converge, say $f(x_n) \to b$ and $f(y_n) \to c$. We need to show that $b = c$. Let $\epsilon > 0$. Since f is uniformly continuous on A, we can choose $\delta > 0$ so that for all $x, y \in A$ we have $|x - y| < \delta \Longrightarrow |f(x) - f(y)| < \frac{\epsilon}{3}$. Since $x_n \to a$ and $y_n \to a$ and $f(x_n) \to b$ and $f(y_n) \to b$, we can choose $n \in \mathbb{Z}^+$ such that $|x_n - a| < \frac{\delta}{2}$, $|y_n - a| < \frac{\delta}{2}$, $|f(x_n) - b| < \frac{\epsilon}{3}$ and $|f(y_n) - c| < \frac{\epsilon}{3}$. Since $|x_n - a| < \frac{\delta}{2}$ and $|y_n - a| < \frac{\delta}{2}$ we have $|x_n - y_n| < \delta$ and hence $|f(x_n) - f(y_n)| < \frac{\epsilon}{3}$. Thus we have $|b - c| \leq |b - f(x_n)| + |f(x_n) - f(y_n)| + |y_n - c| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$. Since $\epsilon > 0$ was arbitrary, we have $|b - c| < \epsilon$ for every $\epsilon > 0$, and hence $b = c$, as required.

Thus we can define $g : \overline{A} \to \mathbb{R}^m$ as follows: given $a \in \overline{A}$ we choose a sequence (x_n) in A with $x_n \to a$, and we define $g(a) = \lim\limits_{n\to\infty} f(x_n)$. Note that when $a \in A$, we do have $g(a) = f(a)$ because we can choose (x_n) to be the constant sequence $x_n = a$ for all a, and then $g(a) = \lim\limits_{n\to\infty} f(x_n) = \lim\limits_{n\to\infty} f(a) = f(a)$.

It remains to show that the map $g : \overline{A} \to \mathbb{R}^m$ is uniformly continuous on $\overline{A}$. Let $\epsilon > 0$. Since f is uniformly continuous on A, we can choose $\delta_1 > 0$ such that for all $x, y \in A$ we have $|x - y| < \delta_1 \Longrightarrow |f(x) - f(y)| < \frac{\epsilon}{3}$. Let $\delta = \frac{1}{3}\delta_1$. Let $a, b \in \overline{A}$ with $|a - b| < \delta$. Choose sequences (x_n) and (y_n) in A with $x_n \to a$ and $y_n \to b$. Since $x_n \to a$ and $y_n \to b$ and $f(x_n) \to g(a)$ and $f(y_n) \to g(b)$, we can choose $n \in \mathbb{Z}^+$ such that $|x_n - a| < \delta$, $|y_n - b| < \delta$, $|f(x_n) - g(a)| < \frac{\epsilon}{3}$ and $|f(y_n) - g(b)| < \frac{\epsilon}{3}$. Then $|x_n - y_n| \leq |x_n - a| + |a - b| + |b - y_n| < \delta + \delta + \delta = \delta_1$ so that $|f(x_n) - f(y_n)| < \frac{\epsilon}{3}$, and hence

$$|g(a) - g(b)| \leq |g(a) - f(x_n)| + |f(x_n) - f(y_n)| + |f(y_n) - g(b)| < \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} = \epsilon.$$