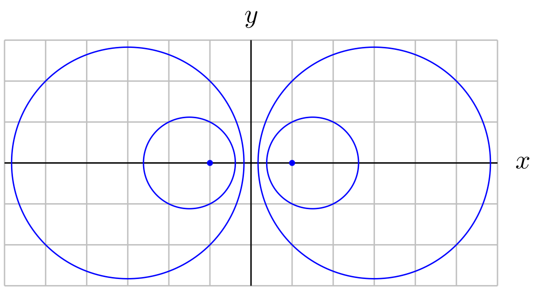
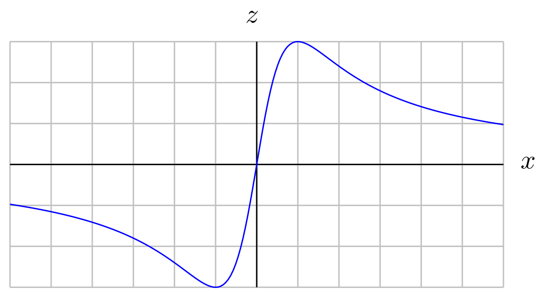
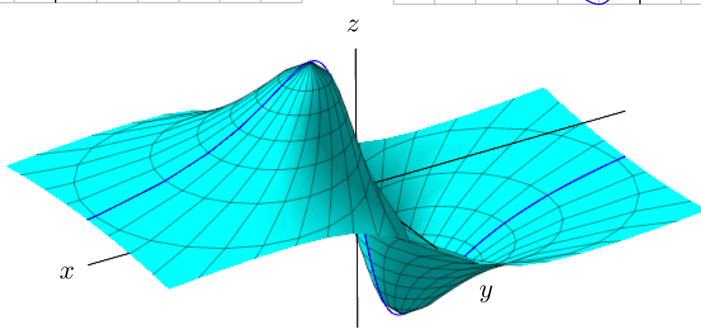
MATH 247 Calculus 3, Solutions to Assignment 1

1: (a) Define $f:\mathbb{R}^2\to\mathbb{R}$ by $z=f(x,y)=\dfrac{6x}{1+x^2+y^2}$. Sketch the level sets $f(x,y)=c$ for $c=0,\pm1,\pm2,\pm3$, and the level set $z=f(x,0)$, then sketch the graph of f, that is sketch the surface $z=f(x,y)$.

Solution: The level set $f(x,y)=0$ is the line $x=0$ (the y-axis). For $c\neq 0$, the level set $f(x,y)=c$ is given by $c\,(1+x^2+y^2)=6x$, that is $x^2-\frac{6}{c}\,x+y^2+1=0$, or $\left(x-\frac{3}{c}\right)^2+y^2=\left(\frac{3}{c}\right)^2-1$. For $0<|c|\leq 3$ this is the circle with center at $\left(\frac{3}{c},0\right)$ and radius $\sqrt{\frac{9}{c^2}-1}$. The surface $z=f(x,y)$ can be drawn by drawing each of the level sets $f(x,y)=c$ at the appropriate height $z=c$. It also helps to sketch the curve $z=f(x,0)=\frac{6x}{1+x^2}$ in the plane $y=0$ (which is the curve of intersection of the surface with the xz-plane). The required level curves $f(x,y)=c$, the curve $z=f(x,0)=\frac{6x}{1+x^2}$, and the surface $z=f(x,y)$ are shown below.

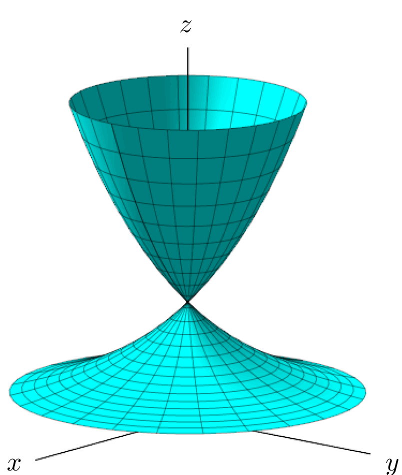






(b) Define $f:\mathbb{R}^2\to\mathbb{R}^3$ by $f(r,\theta)=\big(r\cos\theta,r\sin\theta,e^r\big)$. Sketch the range of f, that is sketch the surface given parametrically by $(x,y,z)=f(r,\theta)$.

Solution: For fixed $r\in\mathbb{R}$, the set of points $(x,y,z)=f(r,\theta)=\big(r\cos\theta,r\sin\theta,e^r\big)$ with $\theta\in\mathbb{R}$ is the circle $x^2+y^2=r^2$, $z=e^r$, that is the circle in the plane $z=e^r$ centered at $(0,0,e^r)$ of radius $|\ln z|$. Letting $c=e^r>0$, so $r=\ln c$, this is the circle in the plane $z=c$ centered at $(0,0,c)$ of radius $|\ln c|$. The image of f is the union of all these circles, so we can sketch the surface by sketching some of these circles for various values of c. (it is the surface of obtained by revolving the curve $z=e^x$ in the xz-plane about the z-axis). Here is the surface:



2: (a) Define $f : \mathbb{R} \to \mathbb{R}^2$ by $f(t) = \big(r(t)\cos t, r(t)\sin t\big)$ where $r(t) = \sin 2t$. Find (with proof) a function $g : \mathbb{R}^2 \to \mathbb{R}$ such that $\text{Range}(f) = \text{Null}(g)$.

Solution: We are being asked to find an implicit Cartesian equation for the curve given in polar coordinates by $r = r(\theta) = \sin 2\theta$. Define $g : \mathbb{R}^2 \to \mathbb{R}$ by $g(x,y) = (x^2+y^2)^3 - 4x^2y^2$. We claim that $\text{Range}(f) = \text{Null}(g)$. Let $(x,y) \in \text{Range}(f)$, say $(x,y) = \big(r\cos t, r\sin t\big)$ with $r = r(t) = \sin 2t = 2\sin t\cos t$. Then we have $x^2+y^2 = r^2 = 4\sin^2 t\cos^2 t$ and hence $(x^2+y^2)^3 = r^6 = 4r^4\sin^2 t\cos^2 t = 4(r\sin t)^2(r\cos t)^2 = 4x^2y^2$ so that $(x,y) \in \text{Null}(g)$. Thus $\text{Range}(f) \subseteq \text{Null}(g)$.

Now let $(x,y) \in \text{Null}(g)$ so we have $(x^2+y^2)^3 = 4x^2y^2$. Case 1: If $(x,y) = (0,0)$ then we can choose $t = 0$ to get $f(t) = (0,0) = (x,y)$. Suppose that $(x,y) \neq (0,0)$ so we have $x^2+y^2 > 0$.

Case 2: If $xy \geq 0$ then choose $t \in \big[0, \frac{\pi}{2}\big] \cup \big[\pi, \frac{3\pi}{2}\big]$ such that $(x,y) = \big(\sqrt{x^2+y^2}\cos t, \sqrt{x^2+y^2}\sin t\big)$. Note that $r(t) = \sin 2t = 2\cos t\sin t = 2\cdot\frac{x}{\sqrt{x^2+y^2}}\cdot\frac{y}{\sqrt{x^2+y^2}} = \frac{2xy}{x^2+y^2}$, and note that since $(x^2+y^2)^3 = 4x^2y^2$ and $xy \geq 0$, we have $(x^2+y^2)^{3/2} = 2xy > 0$ (it is strictly positive because $x^2+y^2 > 0$). Thus

$$f(t) = \big(r(t)\cos t, r(t)\sin t\big) = \Big(\tfrac{2xy}{x^2+y^2}\cdot\tfrac{x}{\sqrt{x^2+y^2}}\,,\ \tfrac{2xy}{x^2+y^2}\cdot\tfrac{y}{\sqrt{x^2+y^2}}\Big) = \Big(\tfrac{2x^2y}{2xy}, \tfrac{2xy^2}{2xy}\Big) = (x,y).$$

Case 3: If $xy \leq 0$ then choose $t \in \big[\frac{\pi}{2}, \pi\big] \cup \big[\frac{3\pi}{2}, 2\pi\big]$ such that $(-x,-y) = \big(\sqrt{x^2+y^2}\cos t, \sqrt{x^2+y^2}\sin t\big)$. Note that $r(t) = \sin 2t = 2\cos t\sin t = 2\cdot\frac{-x}{\sqrt{x^2+y^2}}\cdot\frac{-y}{\sqrt{x^2+y^2}} = \frac{2xy}{x^2+y^2}$, and note that since $(x^2+y^2)^3 = 4x^2y^2$ with $xy \leq 0$ we have $(x^2+y^2)^{3/2} = -2xy > 0$. Thus

$$f(t) = \big(r(t)\cos t, r(t)\sin t\big) = \Big(\tfrac{2xy}{x^2+y^2}\cdot\tfrac{-x}{\sqrt{x^2+y^2}}\,,\ \tfrac{2xy}{x^2+y^2}\cdot\tfrac{-y}{\sqrt{x^2+y^2}}\Big) = \Big(\tfrac{-2x^2y}{-2xy}, \tfrac{-2xy^2}{-2xy}\Big) = (x,y).$$

In all cases, we can find $t \in \mathbb{R}$ such that $(x,y) = f(t)$, and hence $\text{Null}(g) \subseteq \text{Range}(f)$.

(b) Define $g : \mathbb{R}^3 \to \mathbb{R}^2$ by $g(x,y,z) = \big(x^2+y^2-z\,,\ x^2-2x+y^2\big)$, Find (with proof) a function $f : \mathbb{R} \to \mathbb{R}^3$ such that $\text{Range}(f) = \text{Null}(g)$.

Solution: We are being asked to find a parametric equation for the curve of intersection of the two surfaces $x^2+y^2-z = 0$ and $x^2-2x+y^2 = 0$. The first surface $z = x^2+y^2$ is a paraboloid (opening upwards with its vertex at the origin). The equation of the second surface can be rewritten as $(x-1)^2+y^2 = 1$, and so it is a cylinder of radius 1 (with its axis along the vertical line $x = 1, y = 0$). We can parametrize the circle $(x-1)^2+y^2 = 1$ (in the xy-plane) by $(x,y) = \big(1+\cos t, \sin t\big)$. We also need

$$z = x^2+y^2 = (1+\cos t)^2 + (\sin t)^2 = 1 + 2\cos t + \cos^2 t + \sin^2 t = 2 + 2\cos t\,.$$

Thus we shall define $f : \mathbb{R} \to \mathbb{R}^3$ by

$$f(t) = (x,y,z) = \big(1+\cos t, \sin t, 2+2\cos t\big).$$

Let us verify that $\text{Range}(f) = \text{Null}(g)$. Suppose first that $(x,y,z) \in \text{Range}(f)$. Choose $t \in \mathbb{R}$ so that $(x,y,z) = f(t) = \big(1+\cos t, \sin t, 2+2\cos t\big)$. Then we have

$$x^2+y^2-z = (1+\cos t)^2 + (\sin t)^2 - (2+2\cos t) = 1 + 2\cos t + \cos^2 t + \sin^2 t - 2 - 2\cos t = 0, \text{ and}$$
$$x^2-2x+y^2 = (1+\cos t)^2 - 2(1+\cos t) + (\sin t)^2 = 1 + 2\cos t + \cos^2 t - 2 - 2\cos t + \sin^2 t = 0$$

and so $g(x,y,z) = \big(x^2+y^2-z\,,\ x^2-2x+y^2\big) = (0,0)$. This shows that $\text{Range}(f) \subseteq \text{Null}(g)$.

Now suppose that $(x,y,z) \in \text{Null}(g)$ so that $g(x,y,z) = \big(x^2+y^2-z\,,\ x^2-2x+y^2\big) = (0,0)$. Since $x^2-2x+y^2 = 0$ we have $(x-1)^2+y^2 = 1$ and so we can choose $t \in [0,2\pi)$ so that $x-1 = \cos t$ and $y = \sin t$. Since $x^2+y^2-z = 0$ we have

$$z = x^2+y^2 = (1+\cos t)^2 + (\sin t)^2 = 1 + 2\cos t + \cos^2 t + \sin^2 t = 2 + 2\cos t$$

(as calculated above) and so $(x,y,z) = \big(1+\cos t, \sin t, 2+2\cos t\big) = f(t)$. This shows that $\text{Null}(g) \subseteq \text{Range}(f)$.

3: (a) Let $S=\big\{(x,y)\in\mathbb{R}^2\big|y>x^2\big\}$. Prove, from the definition of an open set, that A is open in $\mathbb{R}^2$.

Solution: Let $(a,b)\in S$ so we have $b>a^2$ and hence $\sqrt{b}>|a|$. Let $r=\min\left(\frac{b-a^2}{2}\,,\,\frac{\sqrt{b}-|a|}{2}\right)$. We claim that $B\big((a,b),r\big)\subseteq S$. Let $(x,y)\in B\big((a,b),r\big)$. Note that

$$|x-a|\le\sqrt{(x-a)^2+(y-b)^2}=d\big((a,b),(x,y)\big)<r\le\tfrac{\sqrt{b}-|a|}{2}$$

and similarly

$$|y-b|<r\le\tfrac{b-a^2}{2}.$$

It follows that $|x|-|a|\le|x-a|<\frac{\sqrt{b}-|a|}{2}$ so that $|x|\le\frac{\sqrt{b}+|a|}{2}$ and that $b-y\le|y-b|<\frac{b-a^2}{2}$ so that $y>\frac{b+a^2}{2}$. Note that $0\le\big(\sqrt{b}-|a|\big)^2=b+a^2-2|a|\sqrt{b}$ so we have $2|a|\sqrt{b}\le b+a^2$. It follows that

$$x^2<\left(\tfrac{\sqrt{b}+|a|}{2}\right)^2=\tfrac{b+a^2+2|a|\sqrt{b}}{4}\le\tfrac{b+a^2}{2}<y.$$

Since $y>x^2$ we have $(x,y)\in S$. This shows that $B\big((a,b),r\big)\subseteq S$, as claimed, and so S is open.

(b) Define $f:\mathbb{R}\to\mathbb{R}^2$ by $f(t)=\big(\sin t\,,\,t\,e^t\big)$. Prove that the range of f is not closed in $\mathbb{R}^2$.

Solution: Note that $(1,0)\notin\text{Range}(f)$ because to get $te^t=0$ we need $t=0$, so the only point in Range(f) which lies on the x-axis is the point $f(0)=(0,0)$. We claim that $(1,0)$ is a limit point of Range(f). Let $t_n=\frac{\pi}{2}-2n\pi$ for $n\in\mathbb{Z}^+$. Note that $\sin(t_n)=1$ for all $n\in\mathbb{Z}^+$, and $t_n\to-\infty$ so that (using l'Hôpital's Rule)

$$\lim_{n\to\infty}t_ne^{t_n}=\lim_{t\to\infty}t\,e^t=\lim_{t\to\infty}\frac{t}{e^{-t}}=\lim_{t\to\infty}\frac{1}{-e^{-t}}=\lim_{t\to\infty}-e^t=0.$$

Given $r>0$, since $\lim\limits_{n\to\infty}t_ne^{t_n}=0$ we can choose $n\in\mathbb{Z}^+$ such that $t_ne^{t_n}<r$. Then we have

$$f(t_n)=(\sin t_n,t_ne^{t_n})=(1,t_ne^{t_n})\in B^*\big((1,0),r\big)\cap\text{Range}(f).$$

Thus $(1,0)$ is a limit point of Range(f). Since $(1,0)$ is a limit point of Range(f) and $(1,0)\notin\text{Range}(f)$, it follows that Range$(f)$ is not closed (by Part 2 of Theorem 2.19).

(c) Let A be the set of real numbers $x\in[0,1)$ which can be written in base 3 without using the digit 2, or in other words, let A be the set of real numbers of the form $x=\sum_{k=1}^\infty\frac{a_k}{3^k}$ with each $a_k\in\{0,1\}$. Determine whether A is open or closed (or neither) in $\mathbb{R}$.

Solution: We claim that A is closed. Let A_n be the set of all $x\in[0,1)$ of the form $x=\sum_{k=1}^\infty\frac{a_k}{3^k}$ with $a_1,a_2,\cdots,a_n\in\{0,1\}$ and $a_k\in\{0,1,2\}$ for $k>n$. Note that $x\in A_n$ if and only if $a=b+t$ for some b of the form $b=\sum_{k=1}^n\frac{a_k}{3^k}$ with each $a_k\in\{0,1\}$ and for some t of the form $t=\frac{1}{3^{n+1}}\sum_{k=0}^\infty\frac{a_k}{3^k}$ with each $a_k\in\{0,1,2\}$, or equivalently for some $t\in\big[0,\frac{1}{3^{n+1}}\big]$. Thus A_n is the union of the 2^n closed intervals of the form $\big[b,b+\frac{1}{3^{n+1}}\big]$, where $b=\sum\limits_{k=1}^n\frac{a_k}{3^k}$ with each $a_k\in\{0,1\}$. For example, we have $A_1=\big[0,\frac13\big]\cup\big[\frac13,\frac23\big]=\big[0,\frac23\big]$ and $A_2=\big[0,\frac19\big]\cup\big[\frac19,\frac29\big]\cup\big[\frac13,\frac49\big]\cup\big[\frac49,\frac59\big]=\big[0,\frac29\big]\cup\big[\frac13,\frac59\big]$. Since $A=\bigcap_{n=1}^\infty A_n$ and each set A_n is closed, it follows that A is closed (by Theorem 2.14, which follows easily from Theorem 2.13), as claimed.

We remark that $A=\frac12C=\big\{\frac12x\big|x\in C\big\}$ where C is the famous Cantor set, which is the set of $x\in[0,1]$ which can be written in the form $x=\sum_{k=1}^\infty\frac{a_k}{3^k}$ with each $a_k\in\{0,2\}$. One can prove that C is closed in the same way that we proved that A is closed.

4: (a) Let $A, B \subseteq \mathbb{R}^n$. Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Solution: Since $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$ we have $A \cup B \subseteq \overline{A} \cup \overline{B}$. Since $A \cup B \subseteq \overline{A} \cup \overline{B}$ and $\overline{A} \cup \overline{B}$ is closed, it follows (from Definition 2.15) that $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

Note that for $X, Y \subseteq \mathbb{R}^n$, if $X \subseteq Y$ then every closed set containing Y also contains X, and so $\overline{X} \subseteq \overline{Y}$ (by Definition 2.15). Since $A \subseteq A \cup B$ we have $\overline{A} \subseteq \overline{A \cup B}$. Since $B \subseteq A \cup B$ we have $\overline{B} \subseteq \overline{A \cup B}$. Since $\overline{A} \subseteq \overline{A \cup B}$ and $\overline{B} \subseteq \overline{A \cup B}$ we have $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

(b) Let $A \subseteq \mathbb{R}^n$. Show that $A' = \overline{A}'$ or, in other words, show that A and $\overline{A}$ have the same limit points.

Solution: Note first that if $A \subseteq B$ then we have $A' \subseteq B'$: indeed if $a \in A'$ then given $r > 0$ we have $B^*(a, r) \cap B \supseteq B^*(a, r) \cap A \neq \emptyset$. Since $A \subseteq \overline{A}$, it follows that $A' \subseteq \overline{A}'$. It remains to show that $\overline{A}' \subseteq A'$. Let $a \in \overline{A}'$. Let $r > 0$. We must show that $B^*(a, r) \cap A \neq \emptyset$. Since $a \in \overline{A}'$ we can choose an element $x \in B^*\big(a, \frac{r}{2}\big) \cap \overline{A}$. Since $x \in \overline{A} = A \cup A'$, either we have $x \in A$ or we have $x \in A'$. If $x \in A$ then we have $x \in B^*(a, r) \cap A$ so that $B^*(a, r) \cap A \neq \emptyset$. Suppose that $x \in A'$. Let $s = d(x, a)$ and note that since $x \in B^*\big(a, \frac{r}{2}\big)$ we have $0 < s < \frac{r}{2}$. Since $x \in A'$ we can choose $y \in B^*(x, s) \cap A$. Then we have $y \in A$, and we have $y \neq a$ $\big($since $d(y, x) < s = d(a, x)\big)$, and we have $d(y, a) \leq d(y, x) + d(x, a) < s + \frac{r}{2} < r$, and hence $y \in B^*(a, r) \cap A$ so that $B^*(a, r) \cap A \neq \emptyset$, as required.

(c) Let $A, B \subseteq \mathbb{R}^n$ be disjoint closed sets. Show that there exist disjoint open sets $U, V \subseteq \mathbb{R}^n$ with $A \subseteq U$ and $B \subseteq V$.

Solution: Let A and B be disjoint closed sets in $\mathbb{R}^n$. For each $a \in A$, since $A \cap B = \emptyset$ we have $a \in B^c$, and since B is closed so that B^c is open, we can choose $r_a > 0$ such that $B(a, 2r_a) \subseteq B^c$, that is $B(a, 2r_a) \cap B = \emptyset$. Similarly, for each $b \in B$ we can choose $s_b > 0$ such that $B(b, 2s_b) \subseteq A^c$, that is $B(b, 2s_b) \cap A = \emptyset$.

Let $U = \bigcup_{a \in A} B(a, r_a)$ and $V = \bigcup_{b \in B} B(b, s_b)$. Then U and V are open with $A \subseteq U$ and $B \subseteq V$. We claim that $U \cap V = \emptyset$. Suppose, for a contradiction, that $c \in U \cap V$. Since $c \in U = \bigcup_{a \in A} B(a, r_a)$ we can choose $a \in A$ such that $c \in B(a, r_a)$. Since $c \in V = \bigcup_{b \in B} B(b, s_b)$ we can choose $b \in B$ such that $c \in B(b, s_b)$. If $r_a \leq s_b$ then $d(a, b) \leq d(a, c) + d(c, b) < r_a + s_b \leq 2s_b$ so that $a \in B(b, 2s_b)$, but this contradicts the fact that $B(b, 2s_b) \cap A = \emptyset$. Similarly, if $s_b \leq r_a$ then $d(a, b) < 2r_a$ so that $b \in B(a, 2r_a)$, contradicting the fact that $B(a, 2r_a) \cap B = \emptyset$. Thus $U \cap V = \emptyset$, as claimed.