1: Let $a, n$ and $k$ be positive integers. Suppose that $m \geq 3$ and $\text{gcd}(a, m) = 1$. Show that $a^k + (m - a)^k \equiv 0 \mod m^2$ if and only if $m$ is odd and $k \equiv m \mod 2m$.

2: Find the number of positive integers $k$ such that $k^2 + 2013$ is a square.

3: For each positive integer $n$, let $a_n$ be the first digit in the decimal representation of $2^n$, let $b_n$ be the number of indices $k \leq n$ for which $a_k = 1$, and let $c_n$ be the number of indices $k \leq n$ for which $a_k = 2$. Show that there exists a positive integer $N$ such that for all $n \geq N$ we have $b_n > c_n$.

4: Let $\{a_n\}_{n \geq 1}$ be a sequence of positive real numbers such that $a_n \leq \frac{a_{n-1} + a_{n-2}}{2}$ for all $n \geq 3$. Show that $\{a_n\}$ converges.

5: Let $f(x) = ax^2 + bx + c$ with $a, b, c \in \mathbb{Z}$. Suppose that $1 < f(1) < f(f(1)) < f(f(f(1)))$. Show that $a \geq 0$.

6: Let $E$ be an ellipse in $\mathbb{R}^2$ centred at the point $O$. Let $A$ and $B$ be two points on $E$ such that the line $OA$ is perpendicular to the line $OB$. Show that the distance from $O$ to the line through $A$ and $B$ does not depend on the choice of $A$ and $B$. 
1: Find the number of positive integers $k$ such that $k^2 + 10!$ is a perfect square.

2: Let $f : [0, 1] \to \mathbb{R}$ be continuous. Suppose that $\int_0^x f(t) \, dt \geq f(x) \geq 0$ for all $x \in [0, 1]$. Show that $f(x) = 0$ for all $x \in [0, 1]$.

3: For each positive integer $n$, let $a_n$ be the first digit in the decimal representation of $2^n$, let $b_n$ be the number of indices $k \leq n$ for which $a_k = 1$, and let $c_n$ be the number of indices $k \leq n$ for which $a_k = 2$. Show that there exists a positive integer $N$ such that for all $n \geq N$ we have $b_n > c_n$.

4: Let $p$ be an odd prime. Show that $\left( \frac{2p}{p} \right) \equiv 2 \text{ mod } p^2$.

5: Let $V$ be a vector space over $\mathbb{R}$. Let $V^*$ be the space of linear maps $g : V \to \mathbb{R}$. Let $F$ be a finite subset of $V^*$. Let $U = \{ x \in V \mid f(x) = 0 \text{ for all } f \in F \}$. Show that for all $g \in V^*$, if $g(x) = 0$ for all $x \in U$ then $g \in \text{Span}(F)$.

6: Let $a$, $b$ and $c$ be positive real numbers. Let $E$ be the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ in $\mathbb{R}^3$. Let $u, v, w \in E$ be such that the set $\{u, v, w\}$ is orthogonal. Show that the distance from the origin to the plane through $u$, $v$ and $w$ does not depend on the choice of $u$, $v$ and $w$. 