Solutions to the Special K Problems, 2012

1: Let \( f(x) = x^4 + 2x^3 \). Find the equation of a line which is tangent to the curve \( y = f(x) \) at two distinct points.

Solution: We have \( f'(x) = 4x^3 + 6x^2 \). The tangent line to \( y = f(x) \) at \( x = a \) is given by \( y = l(x) \) where

\[
 l(x) = f(x) + f'(a)(x-a) = (a^4 + 2a^3) + (4a^3 + 6a^2)(x-a) = (4a^3 + 6a^2)x - (3a^4 + 4a^3).
\]

Note that the function \( g(x) = f(x) - l(x) \) has a double root at \( x = a \), indeed

\[
 g(x) = f(x) - l(x) = x^4 + 2x^3 - (4a^3 + 6a^2)x - (3a^4 + 4a^3)
 = (x-a)(x^3 + (a+2)x^2 + (a^2 + 2a)x - (3a^3 + 4a^2))
 = (x-a)^2(x^2 + (2a+2)x + (3a^2 + 4a)).
\]

In order for \( y = l(x) \) to be tangent to the curve \( y = f(x) \) at another point \((b, f(b))\), we need \( g(x) \) to have another double root at \( x = b \). Since \( g(x) \) is monic, it must be of the form \( g(x) = (x-a)^2(x-b)^2 \), so we must have

\[
x^2 + (2a + 2)x + (3a^2 + 4a) = (x-b)^2 = x^2 - 2b + b^2,
\]
and so \( b = -(a+1) \) (1) and \( b^2 = 3a^2 + 4a \) (2). Put \( b = -(a+1) \) into equation (2) to get \( a^2 + 2a + 1 = 3a^2 + 4a \), that is \( 2a^2 + 2a - 1 = 0 \). Thus \( a = \frac{-2 \pm \sqrt{4+8}}{4} = \frac{-1 \pm \sqrt{3}}{2} \) and \( b = -(a+1) \). When \( a = \frac{-1 + \sqrt{3}}{2} \) we have \( b = \frac{-1 - \sqrt{3}}{2} \), and vice versa. Taking \( a = \frac{-1 + \sqrt{3}}{2} \), we have \( a^2 = \frac{-2 - \sqrt{3}}{2} \), \( a^3 = \frac{-5 + 3\sqrt{3}}{4} \) and \( a^4 = \frac{-7 - 4\sqrt{3}}{2} \) and so the equation of the required tangent line is

\[
y = l(x) = (4a^3 + 6a^2)x - (3a^4 + 4a^3) = ((-5 + 3\sqrt{3}) + (6 - 3\sqrt{3}))x - (\frac{21 - 12\sqrt{3}}{4} + \frac{-20 + 12\sqrt{3}}{4}) = x - \frac{1}{4}.
\]

2: Find the area of the region \( R = \{(x, y) \in \mathbb{R}^2 | (x^2 + y^2)^2 \leq 4x^2 \text{ and } x(x^2 + y^2) \leq 2\sqrt{3}xy \} \).

Solution: When \( x > 0 \) we have \( (x^2 + y^2)^2 \leq 4x^2 \iff x^2 + y^2 \leq 2x \iff (x - 1)^2 + y^2 \leq 1 \) and we have \( x(x^2 + y^2) \leq 2\sqrt{3}xy \iff \frac{x^2 + y^2}{2} \leq \sqrt{3}y \iff x^2 + y^2 \leq (y - \sqrt{3})^2 \leq 3 \), and so the part of the region \( R \) which lies to the right of the \( y \)-axis is the region \( A \) which lies inside both the circle centered at \((1,0)\) of radius 1 and the circle centered at \((0, \sqrt{3})\) of radius \( \sqrt{3} \). When \( x < 0 \), on the other hand, we have \( (x^2 + y^2)^2 \leq 4x^2 \iff x^2 + y^2 \leq -2x \iff (x + 1)^2 + y^2 \leq 1 \) and \( x(x^2 + y^2) \leq 2\sqrt{3}xy \iff x^2 + y^2 \geq 2\sqrt{3}y \iff x^2 + (y - \sqrt{3})^2 \geq 3 \) and so the part of the region \( R \) which lies to the left of the \( y \)-axis is the region \( B \) which lies inside the circle centered at \((-1,0)\) of radius 1 and outside the circle centered at \((0, \sqrt{3})\) of radius \( \sqrt{3} \). The area of \( R \) is the sum of the areas of \( A \) and \( B \) which, by symmetry, is equal to the area of a unit circle, namely \( \pi \).
3: Let \( x_n \) be the number of \( 2 \times n \) matrices with entries in \( \{0, 1\} \) which do not contain the \( 2 \times 2 \) block \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

Find \( \lim_{n \to \infty} \frac{x_{n+1}}{x_n} \).

Solution: Let \( a_n, b_n, c_n \) and \( d_n \) be the number of allowable \( 2 \times n \) matrices which end with the column \( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \), \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \). Note that \( a_1 = b_1 = c_1 = d_1 = 1 \). Each of the three columns \( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \), \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) can be appended to any allowable \( 2 \times n \) matrix to get an allowable \( 2 \times (n + 1) \) matrix, so we have

\[
a_{n+1} = c_{n+1} = d_{n+1} = a_n + b_n + c_n + d_n.
\]

It follows that \( a_n = c_n = d_n \) for all \( n \geq 1 \), and we can write the above recursion formula as

\[
a_{n+1} = 3a_n + b_n.
\]

The column \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) can be appended to any allowable \( 2 \times n \) matrix which does not end with \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \), so we have

\[
b_{n+1} = a_n + b_n + d_n = 2a_n + b_n.
\]

From the formula \( a_{n+1} = 3a_n + b_n \) we get \( b_n = a_{n+1} - 3a_n \) (hence also \( b_{n+1} = a_{n+2} - 3a_n \)). Put this into the formula \( b_{n+1} = 2a_n + b_n \) to get \( a_{n+2} - 3a_{n+1} = 2a_n + a_{n+1} - 3a_n \) which we can also write as

\[
a_{n+2} = 4a_{n+1} - a_n.
\]

Note that \( x_n = a_n + b_n + c_n + d_n = 3a_n + b_n = a_{n+1} \), so that \( x_1 = 4 \), \( x_2 = 15 \) and for \( n \geq 2 \) we have

\[
x_{n+1} = 4x_n - x_{n-1}.
\]

Dividing by \( x_n \) gives

\[
\frac{x_{n+1}}{x_n} = 4 - \frac{x_{n-1}}{x_n}.
\]

The above formula shows that \( \left\{ \frac{x_{n+1}}{x_n} \right\} \) is decreasing, and we have \( x_{n+1} = a_{n+2} = 3a_{n+1} + b_{n+1} \geq 3a_{n+1} = 3x_n \) so that \( \frac{x_{n+1}}{x_n} \geq 3 \), and so the sequence \( \left\{ \frac{x_{n+1}}{x_n} \right\} \) must converge with \( \lim_{n \to \infty} \frac{x_{n+1}}{x_n} \geq 3 \). Let \( L = \lim_{n \to \infty} \frac{x_{n+1}}{x_n} \). By taking the limit on both sides of the formula \( \frac{x_{n+1}}{x_n} = 4 - \frac{x_{n-1}}{x_n} \) we obtain \( L = 4 - \frac{1}{L} \), that is \( L^2 - 4L + 1 = 0 \), and so \( L = \frac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{2} \). Since \( L \geq 3 \) we must have \( L = 2 + \sqrt{2} \).

4: Let \( k \geq 3 \) be an integer. Let \( n = \frac{k(k+1)}{2} \). Let \( S \subseteq \mathbb{Z}_n \) with \( |S| = k \). Show that \( S + S \neq \mathbb{Z}_n \).

Solution: Say \( S = \{a_1, a_2, \ldots, a_k\} \). Then each element of \( S + S \) is of the form \( a_i + a_j \) for some 1 or 2-element subset \( \{a_j, a_k\} \subseteq S \) (where we allow the possibility that \( a_j = a_k \)). There are \( \frac{k(k+1)}{2} \) such subsets, and so to show that \( S + S \neq \mathbb{Z}_n \) it suffices to find two distinct sets \( \{a_i, a_j\} \neq \{a_j, a_k\} \) with \( a_i + a_j \) and \( a_i + a_k \).

There are \( k(k-1) \) ordered pairs \( (a_i, a_j) \) with \( a_i \neq a_j \). For such pairs, there are \( n - 1 \) possible values for the difference \( a_i - a_j \) in \( \mathbb{Z}_n \) (since the difference cannot be zero). For \( k \geq 3 \) we have

\[
k(k-1) = \frac{k(k+1)}{2} + \frac{k(k-3)}{2} \geq \frac{k(k+1)}{2} = n > n - 1
\]

so by the Pigeonhole principle, we can choose two order pairs \( (a_i, a_j) \neq (a_k, a_l) \) with \( a_i \neq a_j \) and \( a_k \neq a_l \) such that \( a_i - a_j = a_k - a_l \). Note that \( a_i + a_l = a_j + a_k \) and note that \( \{a_i, a_l\} \neq \{a_j, a_k\} \) (indeed, if we had \( \{a_i, a_l\} = \{a_j, a_k\} \) then since \( a_j \neq a_l \) we would need \( a_i = a_k \), and since \( a_i \neq a_k \) we would need \( a_l = a_j \), but then we would have \( (a_i, a_j) = (a_k, a_l) \)).
5: Let \( f : \mathbb{R} \to \mathbb{R} \). Suppose that \( \lim_{x \to 0} f(x) = f(0) = 0 \) and \( \lim_{x \to 0} \frac{f(2x) - f(x)}{x} = 0 \). Show that \( f \) is differentiable at 0 with \( f'(0) = 0 \).

Solution: Let \( \epsilon > 0 \). Choose \( \delta > 0 \) so that \( 0 < |x| < \delta \implies \left| \frac{f(2x) - f(x)}{x} \right| < \frac{\epsilon}{2} \). Let \( x \in \mathbb{R} \) with \( 0 < |x| < \delta \).

Note that for \( k \in \mathbb{Z}^+ \) we have \( 0 < |x| < \delta \) and so \( \left| \frac{f(x_2^k) - f(x_1^k)}{x} \right| < \frac{\epsilon}{2} \), hence \( \left| \frac{f(x_2^k) - f(x_1^k)}{x} \right| < \frac{\epsilon}{2^{k+1}} \).

Thus for all \( n \in \mathbb{Z}^+ \) we have

\[
\left| \frac{f(x) - f(0)}{x} \right| = \left| \frac{f(x)}{x} \right| = \left| \frac{f(x) - f(x_1^k)}{x} + \frac{f(x_1^k) - f(x_2^k)}{x} + \cdots + \frac{f(x_n^k) - f(x_{n+1}^k)}{x} \right|
\leq \left| \frac{f(x) - f(x_1^k)}{x} \right| + \left| \frac{f(x_1^k) - f(x_2^k)}{x} \right| + \cdots + \left| \frac{f(x_n^k) - f(x_{n+1}^k)}{x} \right| + \left| \frac{f(x_{n+1}^k)}{x} \right|
< \frac{\epsilon}{4} + \frac{\epsilon}{8} + \cdots + \frac{\epsilon}{2^{n+1}} + \left| \frac{f(x_{n+1}^k)}{x} \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2^{|x|}}.
\]

In particular, choosing \( n \) large enough so that \( |f(x_{n+1}^k)| < \frac{\epsilon}{2^{|x|}} \) (which we can do since \( \lim_{x \to 0} f(x) = 0 \)) we have

\[
\left| \frac{f(x) - f(0)}{x} \right| < \epsilon.
\]

6: Let \( \mathbb{Z}^+ \) be the set of positive integers. Show that there exists a bijection \( f : \mathbb{Z}^+ \to \mathbb{Z}^+ \) with the property that \( \prod_{k=1}^n f(k) \) is an \( n \)th power for every \( n \in \mathbb{Z}^+ \).

Solution: We construct such a bijection. We define \( f(1) = 1 \). Having defined \( f(1), f(2), \ldots, f(2n-1) \), we define \( f(2n) \) and \( f(2n+1) \) as follows. First we define \( f(2n+1) \) to be the smallest positive integer with \( f(2n+1) \notin \{f(1), f(2), \ldots, f(2n-1)\} \), and then we define

\[
f(2n) = (f(1)f(2) \cdots f(2n-1))^{(2n)(2n+1)-1} f(2n+1)^{2n}.
\]
1: Find the volume of the solid $S = \{(x,y,z) \in \mathbb{R}^3 \mid (x^2 + y^2 + z^2)^2 \leq 4x^2 \text{ and } x(x^2 + y^2) \leq xz^2 \}$.

Solution: When $x > 0$ we have $(x^2 + y^2 + z^2)^2 \leq 4x^2 \iff x^2 + y^2 + z^2 \leq 2x \iff (x-1)^2 + y^2 + z^2 \leq 1$ and we have $x(x^2 + y^2) \leq xz^2 \iff x^2 + y^2 \leq z^2$, and so the part of the solid $S$ which lies to the right of the $yz$-plane is the region $A$ which lies inside both the sphere centered at $(1,0,0)$ of radius 1 and the double cone $x^2 + y^2 = z^2$. When $x < 0$, on the other hand, we have $(x^2 + y^2 + z^2)^2 \leq 4x^2 \iff x^2 + y^2 + z^2 \leq -2x \iff (x+1)^2 + y^2 + z^2 \leq 1$ and $x(x^2 + y^2) \leq xz^2 \iff x^2 + y^2 \geq z^2$ and so the part of the solid $S$ which lies to the left of the $yz$-plane is the region $B$ which lies inside the sphere centred at $(-1,0,0)$ of radius 1 and outside the double cone $x^2 + y^2 = z^2$. The volume of $S$ is the sum of the volumes of $A$ and $B$ which, by symmetry, is equal to the volume of a unit sphere, namely $\frac{4\pi}{3}$.

2: Find the number of $3 \times n$ matrices with entries in $\{0, 1\}$ which do not contain the $2 \times 2$ block $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Solution: For $k = 0, 1, 2, \ldots, 7$, let $a_{k,n}$ be the number of allowable $3 \times n$ matrices ending with the column which corresponds to the binary representation of $k$. Note that $a_{k,1} = 1$ for all $k$. Since each of the columns $(0,0,0)^T,(1,0,0)^T,(1,1,0)^T,(1,1,1)^T$ can be appended to any allowable $3 \times n$ matrix to obtain an allowable $3 \times (n+1)$ matrix, we have $a_{0,n+1} = a_{4,n+1} = a_{6,n+1} = a_{7,n+1} = a_{0,n} + a_{1,n} + a_{2,n} + \cdots + a_{7,n}$.

Since each of the columns $(0,0,1)^T,(1,0,1)^T$ can be appended to any allowable $3 \times n$ matrix with any final column other than $(0,1,0)^T$ or $(1,1,0)^T$ we have $a_{1,n+1} = a_{5,n+1} = a_{0,n} + a_{1,n} + a_{3,n} + a_{4,n} + a_{5,n} + a_{7,n}$.

Since each of the columns $(0,1,0)^T,(0,1,1)^T$ can be appended to any allowable $3 \times n$ matrix with any final column other than $(1,0,0)^T,(1,0,1)^T$ we have $a_{2,n+1} = a_{3,n+1} = a_{0,n} + a_{1,n} + a_{2,n} + a_{3,n} + a_{6,n} + a_{7,n}$.

We see that $a_{0,n} = a_{4,n} = a_{6,n} = a_{7,n}$ for all $n$, and $a_{1,n} = a_{5,n}$ for all $n$, and $a_{2,n} = a_{3,n}$ for all $n$. Say $a_n = a_{0,n}$, $b_n = a_{1,n}$ and $c_n = a_{2,n}$. Then we have $a_1 = b_1 = c_1$ and the above recursion formulas simplify to $a_{n+1} = 4a_n + 2b_n + 2c_n$

$b_{n+1} = 3a_n + 2b_n + c_n$

$c_{n+1} = 3a_n + b_n + 2c_n$.

By the symmetry between $b$ and $c$ in these equations we see that $b_n = c_n$ for all $n$, so the formulas further simplify to $a_{n+1} = 4a_n + 4b_n$

$b_{n+1} = 3a_n + 3b_n = \frac{3}{4} a_{n+1}$.

Thus we have $a_1 = 1$, $b_1 = 1$, $a_2 = 8$, $b_2 = 7$, and for $n \geq 1$ we have $b_n = \frac{3}{4} a_n$ so that $a_{n+1} = 4a_n + 4b_n = 4a_n + 3a_n = 7a_n$.

Thus for $n \geq 2$ we have $a_n = 8 \cdot 7^{n-2}$ and $b_n = 7 \cdot 7^{n-2}$. For $n \geq 1$, the total number of allowable $3 \times n$ matrices is equal to $4a_n + 4b_n = a_{n+1} = 8 \cdot 7^{n-1}$. 
3: Let $k \geq 3$ be an integer. Let $n = \frac{k(k+1)}{2}$. Let $S \subseteq \mathbb{Z}_n$ with $|S| = k$. Show that $S + S \neq \mathbb{Z}_n$.

Solution: Say $S = \{a_1, a_2, \ldots, a_k\}$. Then each element of $S + S$ is of the form $a_j + a_k$ for some $1 \leq j < k$. There are $\frac{k(k+1)}{2}$ such subsets, and so to show that $S + S \neq \mathbb{Z}_n$ it suffices to find two distinct sets $\{a_i, a_l\} \neq \{a_j, a_k\}$ with $a_i + a_l = a_j + a_k$.

There are $k(k - 1)$ ordered pairs $(a_i, a_j)$ with $a_i \neq a_j$. For such pairs, there are $n - 1$ possible values for the difference $a_i - a_j$ in $\mathbb{Z}_n$ (since the difference cannot be zero). For $k \geq 3$ we have

$$k(k - 1) = \frac{k(k+1)}{2} + \frac{k(k-3)}{2} \geq \frac{k(k+1)}{2} = n > n - 1$$

so by the Pigeonhole principle, we can choose two order pairs $(a_i, a_j) \neq (a_k, a_l)$ with $a_i \neq a_j$ and $a_k \neq a_l$ such that $a_i - a_j = a_k - a_l$. Note that $a_i + a_l = a_j + a_k$ and note that $\{a_i, a_l\} \neq \{a_j, a_k\}$ (indeed, if we had $\{a_i, a_l\} = \{a_j, a_k\}$ then since $a_i \neq a_j$ we would need $a_i = a_k$, and since $a_l \neq a_k$ we would need $a_l = a_j$, but then we would have $(a_i, a_j) = (a_k, a_l)$).

4: Let $f : \mathbb{R}^2 \to \mathbb{R}$. Suppose that $f$ is continuous and that $\int_0^1 f(a + tu) \, dt = 0$ for every point $a \in \mathbb{R}^2$ and every vector $u \in \mathbb{R}^2$ with $|u| = 1$. Show that $f$ is constant.

Solution: Let $a, u \in \mathbb{R}^2$ and with $|u| = 1$. For $x \in \mathbb{R}$, the substitution $t = s + x$ gives

$$\int_x^{1+x} f(a + tu) \, du = \int_0^1 f(a + xu + su) \, ds = 0$$

and so we have

$$\int_0^x f(a + tu) \, dt - \int_x^{1+x} f(a + tu) \, dt = \int_0^1 f(a + tu) \, dt - \int_x^{1+x} f(a + tu) \, dt = 0.$$ 

Differentiate both sides with respect to $x$ using the FTC to get $f(a + xu) - f(a + xu + u) = 0$. In particular, taking $x = 0$, we obtain

$$f(a) = f(a + u).$$

To show that $f$ is constant, we shall show that $f(a) = f(0)$ for all $a \in \mathbb{R}^2$. Given $a \in \mathbb{R}^2$, let $k = |a|$, let $u = \frac{a}{|a|}$ and let $b = a - ku$. Then we have $|b| < 1$ and

$$f(a) = f(a - u) = f(a - 2u) = \cdots = f(a - ku) = f(b).$$

Let $v$ and $w$ be the two points of intersection of the unit circle with the perpendicular bisector of the line segment from 0 to $b$ so that $|v| = |w| = 1$ and $v + w = b$. Then $f(0) = f(v) = f(v + w) = f(b) = f(a).$
5: Let \( \mathbb{Z}^+ \) be the set of positive integers. Show that there exists a bijection \( f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \) with the property that \( \prod_{k=1}^n f(k) \) is an \( n \)-th power for every \( n \in \mathbb{Z}^+ \).

Solution: We construct such a bijection. We define \( f(1) = 1 \). Having defined \( f(1), f(2), \cdots, f(2^n - 1) \), we define \( f(2^n) \) and \( f(2^n + 1) \) as follows. First we define \( f(2^n + 1) \) to be the smallest positive integer with \( f(2^n + 1) \in \{ f(1), f(2), \cdots, f(2^n - 1) \} \), and then we define

\[
 f(2^n) = \left( f(1)f(2)\cdots f(2n-1) \right)^{(2n)(2n+1)^{-1}} f(2n+1)^{2n}.
\]

6: Let \( A \) be an \( n \times n \) matrix. Let \( u \) be an eigenvector of \( A \) for the eigenvalue 1. Suppose that all of the entries of \( A \) and all of the entries of \( u \) are positive. Show that the eigenspace for the eigenvalue 1 is 1-dimensional.

Solution: Let \( v \) be any eigenvector for the eigenvalue 1. We must show that \( u = cv \) for some \( 0 \neq c \in \mathbb{R} \). Suppose that \( v \) has at least one positive entry (otherwise replace \( v \) by \( -v \)). Choose \( k \) with \( v_k > 0 \) to minimize \( \frac{u_k}{v_k} \) (so we have \( \frac{u_k}{v_k} \leq \frac{u_i}{v_i} \) whenever \( v_i > 0 \)). We claim that \( u = \frac{u_k}{v_k} v \). Consider the vector \( w = u - \frac{u_k}{v_k} v \). The \( i \)-th entry of \( w \) is \( w_i = u_i - \frac{u_k}{v_k} v_i \). If \( v_i \leq 0 \) then we have \( w_i \geq u_i > 0 \), and if \( v_i > 0 \) then we have

\[
 w_i = \left( \frac{u_i}{v_i} - \frac{u_k}{v_k} \right) v_i \geq 0,
\]

so we have \( w_i \geq 0 \) for all \( i \). Also note that

\[
 Aw = A\left(u - \frac{u_k}{v_k} v \right) = Au - \frac{u_k}{v_k} Av = u - \frac{u_k}{v_k} v = w.
\]

Suppose, for a contradiction, that \( w \neq 0 \). Then each entry \( w_i \geq 0 \) and some entry \( w_l > 0 \). Since every entry of \( A \) is positive, it follows that every entry of \( Aw \) is positive, indeed the \( i \)-th entry of \( Aw \) is

\[
 (Aw)_i = \sum_{j=1}^n A_{i,j}w_j \geq A_{i,l}w_l > 0.
\]

Since \( w = Aw \), every entry of \( w \) is positive. But this is not possible since \( w_k = u_k - \frac{u_k}{v_k} v_k = 0 \). Thus \( w = 0 \) and so we have \( u = \frac{u_k}{v_k} v \), as claimed.