1: Let $x > 1$ be a real number, and $n > 1$ be an integer. Prove that

$$\sqrt[n]{x} < 1 + \frac{x - 1}{n}.$$ 

2: Find the smallest (by area) right-angled triangle with integral sides in which a square with integral sides can be inscribed so that an angle of the square coincides with the right angle of the triangle.

3: Let $S$ be a set of points in the plane. A circle $C$ is said to be framed by $S$ if $C$ has a diameter whose endpoints both lie in $S$. Find all sets $S$ of four points in the plane such that, for any two circles $C_1$ and $C_2$ framed by $S$, the set $S \cap C_1 \cap C_2$ is non-empty.

4: Let $f$ be a real-valued continuous function of a real variable with the property that

$$\lim_{x \to +\infty} f(f(x)) = +\infty \quad \text{and} \quad \lim_{x \to -\infty} f(f(x)) = -\infty.$$ 

Prove that $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ both exist and are infinite.

5: Peter tells Ian and Christopher that $x$ and $y$ are two integers with $1 < x < y$ and $x + y \leq 30$. Peter then gives Christopher the value of $x + y$ and Ian the value of $xy$.

(1) Ian says “I don’t know the values of $x$ and $y$.”
(2) Christopher replies “I knew that you didn’t know their values.”
(3) Ian responds “I still don’t know the values of $x$ and $y$.”
(4) Christopher exclaims “In that case, I know their values!”

What is the value of $xy$?
1: Define \( Q_k = \sum_{n=1}^{\infty} \frac{1}{(k+n)!} + \frac{2}{(k+n+3)!} + \frac{3}{(k+n+4)!} + \cdots \). Show that \( Q_0 \) is rational, but that \( Q_k \) is irrational for every positive integer \( k \).

2: Let \( S \) be a set of points in the plane. A circle \( C \) is said to be framed by \( S \) if \( C \) has a diameter whose endpoints both lie in \( S \). Find all sets \( S \) of four points in the plane such that, for any two circles \( C_1 \) and \( C_2 \) framed by \( S \), the set \( S \cap C_1 \cap C_2 \) is non-empty.

3: Let \( f \) be a real-valued continuous function of a real variable with the property that
\[
\lim_{x \to +\infty} f(f(x)) = +\infty \quad \text{and} \quad \lim_{x \to -\infty} f(f(x)) = -\infty.
\]
Prove that \( \lim_{x \to +\infty} f(x) \) and \( \lim_{x \to -\infty} f(x) \) both exist and are infinite.

4: Let \( a \) and \( b \) be non-zero complex numbers which satisfy the equation
\[ a \left( 2^{\lfloor a \rfloor} + 2^{\lfloor b \rfloor} \right) = (a + b) \left( 2^{\lfloor a + b \rfloor} \right). \]
Prove that \( a^6 = b^6 \).

5: Find the value of the infinite product \( \prod_{n=1}^{\infty} \left( 1 + \frac{1}{a_n} \right) \) where \( a_1 = 1 \) and \( a_n = n(a_{n-1} + 1) \) for all \( n \geq 2 \).