1: Show that if a real valued function $f$ verifies
\[ f(x + y) = f(xy) \]
for all (strictly) positive real numbers $x$ and $y$, then $f$ is constant over $(0, \infty)$.

2: A list is made of all subsets of the set $S = \{1, 2, \ldots, n\}$, including $S$ and the empty set in the list. Subsets $A_1, A_2, \ldots, A_r$, $r > 1$, are chosen at random from the list (a subset can be chosen more than once). Find the probability that the chosen subsets are pairwise disjoint (i.e. $A_i \cap A_j = \emptyset$ for all $1 \leq i < j \leq r$).

3: Let $P$ be a polynomial on the real numbers. Show that
\[ |P(k) - 3^k| < 1 \text{ for all } k = 0, 1, 2, \ldots, n \]
implies that the degree of $P$ is not less than $n$.

4: A set of $2n+2$ 2-vectors $V_1, V_2, \ldots, V_{2n+2}$ is made by selecting the entries arbitrarily from the set $\{1, 2, 4, 8, \ldots, 2^n\}$. Show that there exists a pair of vectors $V_i, V_j, i \neq j$, such that the $2 \times 2$ matrix
\[ A_{ij} = \begin{pmatrix} V_i \\ V_j \end{pmatrix} \]
has determinant zero.

5: Let $\triangle ABC$ be any triangle and $A', B', C'$ points on sides $BC$, $CA$ and $AB$, respectively, such that the circles inscribed in triangles $\triangle AC'B'$, $\triangle BA'C'$, $\triangle CB'A'$ have equal radii $r$. Let $\overline{r}$ be the radius of the circle inscribed in $\triangle A'B'C'$ and $R$ that of the circle inscribed in $\triangle ABC$. Prove that
\[ R = r + \overline{r}. \]
1: A list is made of all subsets of the set $S = \{1, 2, \cdots, n\}$, including $S$ and the empty set in the list. Subsets $A_1, A_2, \cdots, A_r, r > 1$, are chosen at random from the list (a subset can be chosen more than once). Find the probability that the chosen subsets are pairwise disjoint (i.e. $A_i \cap A_j = \emptyset$ for all $1 \leq i < j \leq r$).

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3: A set $K$ in the plane is said to be Valentine convex if for any set $\{x, y, z\}$ of points in $K$ one of the three line segments $xy, yz, zx$ is contained in $K$.

(a) Show that the union of any two convex sets is Valentine convex. Show that there exist three convex sets whose union is not Valentine convex.

(b) Give an example of a Valentine convex set which cannot be expressed as the union of two convex sets.

4: A figure eight is a closed curve that intersects itself exactly once. Show that any collection of disjoint figure eights in the plane must necessarily be countable.

5: Let $f : \mathbb{R}^+ \to \mathbb{R}$ be twice continuously differentiable. such that

(a) $f(0) = f'(0) = 0$,

(b) $0 \leq 3 f''(x) \leq \sqrt{1 + (f'(x))^2} \left( \cos(f(x)) + 2 \right)$ for all $x > 0$.

Prove that $0 \leq f(x) \leq \cosh x - 1$ for all $x > 0$. 