Gauss

And

Non-Euclidean Geometry

Crash Course Notes
by Stan Burris
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BRIEF TIMELINE

1796 – 1798 **Gauss and Bolyai (senior)** are both studying in Goettingen; they were the best of friends. Gauss left Goettingen in September of 1798, Bolyai in June of 1799. They had a farewill reunion in May, 1799. They never saw each other again.

1816 – 1823 **Lobaschewsky** gives up on the possibility of proving the parallel axiom

1823 **Bolyai (junior)** tells his father of his creation of non-Euclidean geometry. (His father had tried to discourage him from pursuing the study of parallels. He spends several years preparing this; it will appear as an appendix to his father's book.

1831 Reprints of the **Bolyai appendix** are ready. A copy is sent to Gauss, but due to a cholera epidemic only the cover letter from Bolyai (senior) arrives. The copy of the appendix is returned to the sender, Bolyai.

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1832 **Gauss receives the Bolyai appendix** early in the year and sends his reply to Bolyai (senior) in March.

1829 – 1838 **Lobaschewsky** publishes on non-Euclidean geometry in Russian journals. A somewhat inadequate summary appears in Crelle's journal in 1837.

1840 **Gauss** obtains a copy of **Lobaschewsky's 1838 memoir** on non-Euclidean geometry.

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Some Comments on Geometry

- Euclid's geometry was meant to be **the** geometry of the real world.
- In Gauss's time the parallel axiom was considered to be the only axiom of Euclid that was not self evident. The topic of proving the parallel axiom from the other axioms was in the air – such a proof would show that Euclid's geometry was indeed the geometry of the real world, or, as Gauss liked to say, (Euclid's) **geometry is true**.
- For mathematicians the way to prove Euclid's geometry true was to assume the negation of the parallel axiom and derive a contradiction.
- By 1799 Gauss (age 22) has worked out enough of the consequences of the failure of the parallel axiom without finding a contradiction that he says his investigations "make the truth of geometry dubious". By this he surely means that the geometry of Euclid may not be the geometry of the real world.

Some Questions about Non-Euclidean Geometry

- What did it mean **to discover** non-Euclidean geometry?
- Was it merely the **assertion** that the failure of the parallel axiom **might not** lead to a contradiction? Was publication necessary to be credited?
- Or, was it merely the **assertion** that the failure of the parallel axiom **does not** lead to a contradiction? Again, was publication necessary?
- Surely Gauss, Bolyai, Lobaschewsky (and other mathematicians?) who made the latter claim before 1850 were only saying that they had sufficient experience with working out the consequences of assuming the failure of the parallel axiom that they felt confident that they would not encounter a contradiction.
- Saccheri worked out numerous consequences of the failure of the parallel axiom before 1800, but unfortunately concluded that this led to an absurdity. So he is not credited with discovering non-Euclidean geometry.
- Surely Gauss would have published if he had come to the conclusion that Euclidean geometry was not the true geometry? Was the mere 'possibility of an alternative geometry for the true geometry' insufficient reason for him to publish and risk having to deal with critics? Was he interested in any geometry except the true geometry?

From Gauss's WERKE, Volume VIII

1. Gauss to Bolyai (senior) Helmstedt, 16 December, **1799**

[This was written one year after Gauss had finished his studies in Goettingen. This is his first letter to Bolyai after Bolyai had left Goettingen. Evidently Bolyai has communicated to Gauss his claim that he has put Euclidean geometry on solid ground.]

… I am sorry that I didn't use our former close proximity to learn more about your work on the first principles of geometry; I would surely have spared myself considerable wasted effort and have become more tranquil, insofar as this is possible for someone like me when there is so much to be desired in this *[geometry]* situation. I myself have moved far ahead in my work on this (considering that my other heterogeneous tasks leave little time); the path that I have hammered out does not so much lead to the goal that one hopes for, and which you have secured, but much more it makes the truth of geometry dubious. To be sure, I have found much that would qualify as a proof for most *[that Euclidean geometry is correct]*, but which in my eyes really proves NOTHING; for example, if one could prove that a straight-edged triangle exists whose area would be greater than that of a given region, then I would be in the position to rigorously justify the whole of geometry. Most people would accept the former as an axiom; not me; it could be possible that no matter how far apart the vertices of a triangle are assumed to be, still the area would remain under a given bound. I have several such results, but in none of them can I find anything satisfactory. Make your work known soon; for this you will certainly harvest not only the thanks of the general public --- to which many belong who consider themselves sophisticated mathematicians; I become ever more convinced that the number of real mathematicians is extremely small, and most of them can neither judge nor even understand the difficulties of such a work --- but also the thanks of all those whose opinion you value.

In Braunschweig there is an emigrant named Chuvelot, not a bad mathematician, who claims to have completed the work on geometry, and he will soon publish this; but I expect nothing from him. In Hindenburgs Archiv, 9th issue, one finds a new attempt with the same goal, from a certain Hauff, which according to all the reviews …

2. Gauss to Bolyai (senior) Braunschweig, 25 November **1804**

[Bolyai has communicated his proof to Gauss.]

I read through your manuscript with great interest and care, and thoroughly enjoyed the underlying precision. However you don't want my praise which would appear biased because your development of ideas has so much in common with my own. I have sought the solution to this Gordian knot, and till now have sought in vain. You desire only my careful and unfettered judgment: it is that your explanation does not satisfy me. I will try to explain the issue (it belongs to the same set of reefs on which my attempts have run aground) with as much clarity as possible. To be sure, I still have hope that, before my time is up, these reefs will permit passage. For the time being I have so many other tasks at hand that I cannot think about this; believe me, it would really make me happy if you were to pull ahead of me and overcome all obstacles. I would then undertake with the greatest joy, with all that is in my power, to defend your accomplishment and bring it to the light of day. Now I come to the issue at hand.

I find no real fault with any conclusion except one. What doesn't convince me is simply the reasoning in Article XIII. *[Gauss spells out the details of his objection.]*

You demanded my honest opinion and I have given it; and I repeat once again the assurance that it would give me joy if you can overcome all obstacles.

3. Schumacher, November **1808**

[Schumacher was in Goettingen during the winter 1808-1809 and kept a notebook titled Gaussiana in which one finds the following.]

{Gauss has studied the theory of parallels to the point that if the commonly accepted theory is not true then there must be a constant that precedes the choice of a line segment for length, which is absurd. But he *[Gauss]* does not consider the issue *[of geometry]* settled with this.}

4. Schumacher, 27 April **1813**

[This was written on a slip of paper.]

In the theory of parallels we are no further than Euclid. This is the partie hontuese of mathematics that sooner or later must assume a quite different form.

5. Gauss to Gerling, Goettingen, 11 April **1816**

[Gerling studied in Goettingen, and wrote Gauss in March of 1816 concerning Legendre's theory of parallels in the book elemens de geom., 6th edit., 1806. Gauss responds that Legendre's argument does not carry the weight of proof for him, and then comments on what happens if Euclidean geometry is not correct.]

It is easy to show that if Euclid's geometry is not the true one then there are no similar figures: the angles in an equilateral triangle depend on the size of the edges, in which I do not find anything absurd. Then the angle is a function of the side, and the side a function of the angle, naturally such a function in which a linear constant appears. It seems somewhat paradoxical that a linear constant can be a priori possible; but I don't find anything contradictory in this. It would be even desirable that Euclid's geometry is not true, for then we would have a general measure a priori, for example, one could assume as the unit of space the side of the equilateral triangle whose angle = 59^{o} 59' 59''.99999….

6. Gauss Goettingen, 20 April **1816**

[The following is taken from a review that Gauss published in the Scholarly Notices of Goettingen. It concerns two recent booklets purporting to justify Euclidean geometry. The first is by J.C. Schwab and is heavily philosophical (65 pp.) The second is by the mathematics professor Matthias Metternich (44 pp.)]

There are few topics in mathematics on which so much has been written as the gaps occurring in the foundation of the theory of parallel lines at the beginning of geometry. Seldom does a year pass without a new attempt to fill these gaps, and without our being able to say, if we speak honestly and openly, that we have in any way gone beyond where Euclid was 2000 years ago. Such a proper and direct admission seems more becoming of the honor of science than the frivolous efforts to hide the gaps that one cannot fill with an untenable web of apparent proofs.

The author of the first work made a similar attempt 15 years ago in a little work "Tentamen novae parallelarum theoriae notione situs fundamentae" in which he tried to base everything on the concept of identity of direction *[Lage --- perhaps this would be better translated by 'how it lies in space']*. He defined parallel lines as lines with the same direction and from this concludes that such lines must necessarily cut a third line in the same angle because this angle is nothing more than the difference in direction of the third line with that of the parallel lines. This method of proof is repeated in his new work without one being able to conclude that anything has been gained from the interwoven philosophical observations.

….The assertion on page 24 "Notionem situs e geometria" … must be rejected by every geometer, given the concept of direction used by the author in his proofs.
[Gauss goes on to say that a big part of the booklet concerns the work of Kant; work which Gauss says is, at least in part, misconstrued by Schwab.]

Although the author of the second work treats his subject in a quite different and truly mathematical manner, nonetheless we cannot come to a more favourable conclusion regarding his results. *[Gauss proceeds in the space of 2 pages to point out the key error concerning a sequence of points on a line monotonely moving towards a given point: it does not follow that the sequence approaches the point as a limit.]* It is barely comprehensible that the author could so deceive himself…. It seems that a grammatical ambiguity led the author to error, namely the ambiguity of '<u>a</u> given magnitude'……

We must almost regret having dwelled so long on such a well known and simple matter, whose author however appears to have truly thought correct, for prior to publication this matter, in published pages, had been clearly brought to his attention ….

7. Gauss to Olbers Goettingen, 28 April **1817**

Wachter has printed off a small piece about the first principles of geometry of which you will receive a copy from Lindenau. Although Wachter has penetrated into the matter more than his predecessors, still his proof is no more binding than all the others. I am coming ever more to the conviction that the necessity of our geometry cannot be proved, at least not by human comprehension nor for human comprehension. Perhaps in another life we will come to other views on the nature of space which are currently unobtainable for us. Until then one must not put Geometry into the same rank as Arithmetic, which stands a priori, but rather in the same rank as, say, Mechanics.

8. Gerling to Gauss Marburg, 23 July, **1818**

[Gerling has been given the job of preparing a new edition of Lorenz's pure mathematics and he is concerned about how to present the geometry. So he writes Gauss for advice.]

….For the geometry I have fewer problems, but I would still like to ask how best to present the parallels theory. What Lorenz has is partly false, partly unfounded. I find it correct to present the Euclidean approach, but pointing out the shortcomings….I thought it best to present the assertion 'a line can have only one parallel through a point' as an axiom, and in a commentary say that no proof for this has as of yet been found, and thus until one is found, or the incorrectness of the assertion is proved, we must take it as an axiom, just as in principle Euclid did. Please give me your opinion on this….

9. Gauss to Gerling Goettingen, 25 August **1818**

…I am happy that you have the courage to express yourself as if you recognized the possibility that our parallels theory along with our entire geometry could be false. But the wasps whose nest you disturb will fly around your head ….

10. Gerling to Gauss Marburg, 25. January, **1819**

[Gerling is happy to proceed with the revision of Lorenz's work on pure mathematics pointing out the dubious nature of the theory of parallels. He also wants to inform Gauss of another convert, Schweikart.]

{…The part about the theory of parallels I have formulated as follows…. 'This proof (the parallels theory) has been tackled in many ways by clever mathematicians, but until now nothing completely satisfactory has been found. As long as this is lacking the assertion, along with all that depends on it, remains a hypothesis whose validity for our everyday life is clearly supported by experience, but whose general necessary correctness can be doubted without absurdity.'

While speaking of the theory of parallels I must tell you something, and fulfill an obligation. I found out in the previous year that my colleague Schweikart (Professor of Law, now the Prorector) has occupied himself considerably with mathematics, namely with parallels. I asked him to loan me his book. While promising me this he told me that he now saw that his book (1808) had mistakes (he had assumed for example 4-sided figures with all angles equal as a fundamental concept), but that he had not given up his work on this subject, and now was almost convinced that without some data the Euclidean theorem could not be proved, and that it didn't seem improbable to him that our geometry was just a chapter in a more general setting. I told him how you had publicly stated several years ago that since the time of Euclid no progress had been made; indeed that you had often told me that by varied approaches to this topic you were also unable to show the absurdity of such an assumption. When he sent me the requested book the enclosed note was attached and he asked me shortly thereafter (end of December) to include it with this letter, and to ask you as a request from him to give your opinion on his ideas…..}

The next item is the note of Schweikart mentioned in this letter.

11. Schweikart's Note to Gauss Marburg, December **1818**

[Schweikart introduces the name Astral Geometry for his non-Euclidean geometry.]

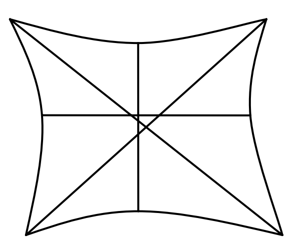
{There is a two-fold geometry, a geometry in the narrow sense, the Euclidean; and an astral study of magnitudes.

The triangles of the latter have the property that the sum of the three angles of a triangle is not equal to two right angles.

With this assumption the following can be rigorously proved:

a) the sum of the 3 angles in a triangle is less than two right angles;
b) the sum gets smaller as the area of the triangle increases;
c) the height of a isosceles right triangle increases as the side gets longer, but it cannot increase beyond a certain line that I call the constant.

Squares have the following shape *[ed: the drawing should be symmetric, with all lines crossing in the center]*:



If for us this constant is half the earth axis (from which it follows that every line drawn from one fixed star to another, being 90^o apart, would be tangent to the earth), then it is infinitely large in relation to the space of things occurring in everyday life.

The Euclidean geometry holds only under the assumption that the constant is infinitely large. Only then is it true that the three angles of a triangle are equal to two right angles; also this is easily proved, given the assertion that the constant is infinitely large.

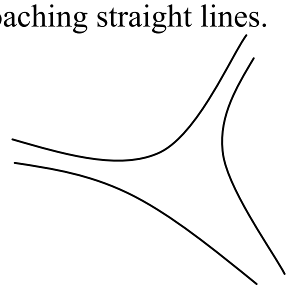
12. Gauss to Gerling Goettingen, 16. March **1819**

….The note of Herr Professor Schweikart gave me an incredible amount of pleasure, and I ask you to convey to him my very best. For me it was everything straight from the soul. It is just that with the one part that begins:

"If for us this constant is the half of the earth axis", and so forth, I must make three comments:

1) I don't see the possibility that one constant can be valid just for us, and for other beings another. I don't know if Hr. Sch. meant it this way, it is just that he himself underlined for us.
2) He continues: "whereby every line segment drawn from one fixed star to another, being 90^o apart, would be tangent to the earth". In this case the distance of the fixed star compared to the constant is immeasurably large; however with this in mind the angle of 90^o has a definite sense only with respect to a fixed vertex, say the center of the earth, which without doubt Hr. Prof. Sch. tacitly assumed.
3) Hr. Prof. Sch. doubtless gave this merely as an example to illustrate, for although I can really imagine that the Euclidean geometry is not correct, nonetheless from our astronomical experiences the constant must be immeasurably greater than the radius of the earth.

I suspect that Hr. Sch. will be in agreement with all of these points, which would make me very happy that his view is in complete agreement with my own. I only want to point out that I have developed the astral geometry so far that I can completely solve all problems once the constant = C is given. The defect of the angle sum in the plane triangle from 180^o is, for example, not just greater as the area gets greater, but it is exactly proportional to it, so that the surface area has a bound that it can never reach, and this bound equals the area between three asymptotically approaching straight lines.



The formula for this bound is: Limes areae trianguli plani = $$\frac{\pi CC}{\{\log hyp(1+\sqrt{2})\}^2}$$

Also every other polygon with a fixed number of sides = n has as concerns its area a definite bound which it can approach arbitrarily close but not reach, *[= above times (n-2)]*.

13. Gauss to Taurinus Goettingen 8 November **1824**

[According to Dunnington, Taurinus, a lawyer like his uncle Schweikart, was stimulated by his uncle's book to work on the theory of parallels. In October 1824 he sent Gauss his proof of the parallel axiom, and hence that geometry is Euclidean.]

I read your nice letter of 30 Oct. along with the attached short article, not without pleasure, even more so as otherwise I am accustomed to find no trace of true geometric insight among the majority of those who undertake new attempts regarding the so-called theory of parallel lines. I have nothing to say (or not much) against your attempt except that it is incomplete. To be sure your presentation of the proof that the sum of the three angles of a triangle cannot exceed 180^o needs some sharpening in geometric rigor. This should be done, and there is no doubt that this impossibility can be rigorously proved. It is quite a different matter with the 2^{nd} part, that the sum of the angles cannot be smaller than 180^o; this is the real knot, the reef on which everything runs aground. I suspect that you have not dealt with this topic very much. For me it has been more than 30 years, and I doubt that anyone else has been more involved with this. The assumption that the sum of the three angles is smaller than 180^o leads to a geometry that is quite different from ours (Euclidean), which is consistent *[in sich selbst durchaus consequent ist]*, and which I have developed quite satisfactorily to the point that I can resolve every question in it with the exception of the determination of a constant which does not present itself a priori. As one takes this constant ever larger the more this geometry approaches Euclidean geometry, and an infinite value makes the two agree. The theorems of this geometry appear in part paradoxical and, to the inexperienced, nonsensical. However with careful and calm reflection they contain nothing impossible. Thus, for example, the three angles of a triangle can be made as small as desired if one takes the sides large enough; on the other hand the area of a triangle, no matter how large the sides are, cannot exceed a certain bound, nor ever reach it. All of my efforts to find a contradiction, an inconsistency *[Inconsequenz]* in this non-Euclidean geometry have been fruitless; the only thing in it that contradicts our comprehension is that if it were true then there must be a linear magnitude in space (although we don't know it). But it seems to me that in spite of the word-mastery of the metaphysicians, we know really too little, or even nothing at all, about the true nature of space to be able to confuse something that seems unnatural with absolutely impossible. If non-Euclidean geometry is the real one and the constant is comparable to the magnitudes that we encounter on earth or in the heavens then it can be determined a posteriori. I have therefore occasionally for fun expressed the wish that Euclidean geometry not be the real one, for then we would have a priori an absolute measure.

From one who has described me as a thinking mathematical head I need not fear that the above will be misunderstood: at any rate you have this as a private communication that is in no way to be used publicly or in a publication. Perhaps I myself will make these investigations known in the future if I can find more leisure than in my present situation.

According to **Dunnington, p. 182:**

"This letter stimulated the young Taurinus to continue his research with increased zeal. In 1825 he published his *Theory of Parallel Lines* in which he was convinced of the unconditional validity of the parallel axiom, but he began to develop the results which are yielded by a rejection of it. Thus he arrived at that constant which would be peculiar to non-Euclidean geometry. In the simultaneous possibility of infinitely many such geometries, each of which is without internal contradiction, he saw sufficient reason to reject all of them."

14. Gauss to Bessel Goettingen 27 January **1829**

…There is another topic, one which for me is almost 40 years old, that I have thought about from time to time in isolated free hours, I mean the first principles of geometry; I don't know if I have ever spoken to you about this. Also in this I have further consolidated many things, and my conviction that we cannot completely establish geometry a prioir has become stronger. In the meantime it will likely be quite a while before I get around to preparing my very extensive investigations on this for publication; perhaps this will never happen in my lifetime since I fear the cry of the Boetians if I were to voice my views. It is strange, however, that except for the well known gaps in Euclid's geometry which till now one has tried in vain to fill, and never will fill, there are other defects in the subject that to my knowledge no one has touched, and to resolve these is by no means easy (but possible). Such is the definition of a plane as a surface for which the line joining any two of its points lies wholly in it. This definition contains more than is necessary for the description of the surface, and tacitly involves a theorem which must be proved first ….

15. Bessel to Gauss Koenigsberg 10 February **1829**

{… I would protest loudly if you were to allow "the cry of the Boetians" to thwart the working out of your geometry views. From what Lambert has said, and what Schweikart told me, it has become clear that our geometry is incomplete and needs a correction which is hypothetical and which disappears if the sum of the angles of a triangle = 180^o. The latter would be the real geometry, the Euclidean one, which practically, at least for figures on the earth …..}

16. Gauss to Bessel Goettingen 9 April **1830**

…The ease with which you delved into my views on geometry gives me real joy, given that so few have an open mind for such. My innermost conviction is that the study of space is a priori completely different than the study of magnitudes; our knowledge of the former is missing that complete conviction of necessity (thus of absolute truth) that is characteristic of the latter; we must in humility admit that if number is merely a product of our minds, space has a reality outside our minds whose laws we cannot a priori state …

17. Gauss to Gerling Goettingen 14 February **1832**

…In addition I note that in recent days I received a small work from Hungary on non-Euclidean geometry in which I find all of my ideas and **results** developed with great elegance, although in a concentrated form that is difficult for one to follow who is not familiar with the subject. The author is a very young Austrian officer, the son of a friend of my youth with whom I had often discussed the subject in 1798, although my ideas at that time were much less developed and mature than those obtained by this young man through his own reflections. I consider this young geometer, v. Bolyai, to be a genius of the first class….

18. Gauss to Bolyai (senior) Goettingen 6 March **1832**

[The original of this letter is lost. This is from a copy made by Bolyai (junior) and sent by his father to Sartorius on 26 August 1856.]

….Now for some remarks about the work of your son.

If I start by saying "I cannot praise it" then you will most likely be taken aback; but I cannot do otherwise; to praise it would be to praise myself; the entire contents of the work, the path that your son has taken and the results to which it leads, are almost perfectly in agreement with my own meditations, some going back 30 – 35 years. In truth I am astonished. My intention was not to release any of my own work in my lifetime. Most people don't have a true sense of what is involved, and I have found very few who are particularly interested. To appreciate what is going on one must first of all have a real grasp of what is missing, and on this point most are in the dark. On the other hand it was my intention to write everything down so that it didn't perish with me.

So I am truly surprised that I am now spared this effort, and it is the greatest joy for me that precisely the son of my old friend is the one who preceded me in such a remarkable manner.

[Gauss then spends 2 ½ pages suggesting notation and sketching his own proofs of some of the results.]

I have merely given an outline of the proofs, without details and polish, as I have no time to devote to this. Feel free to communicate these to your son; at least I ask you to give him my best regards and assure him of my particularly high esteem; at the same time urge him to work on the following problem:

"Determine the volume of the tetrahedron"

Since the area of a triangle is so easy to find one would expect that the volume would have an equally easy expression; but this expectation is not fulfilled.

In order to properly treat geometry from the beginning it is necessary to prove the existence of a plane; the usual definitions are excessive, already subsuming a theorem. One must wonder at the fact that all the authors of Euclidean geometry up to the most recent times have gone about their work so carelessly; this difficulty is of a completely different nature from that of deciding between Σ and S; the former is not at all difficult to remove. On this matter I will likely find myself quite satisfied with the treatment in your book.

Precisely the impossibility of deciding a priori between Σ and S gives the clearest proof that Kant was not justified in asserting that space is <u>just the form</u> of our perception. Another equally strong reason is in a brief essay in the Scholarly Notices of Goettingen 1831, article 64, p. 65 …..