MY GEOMETRY HISTORY LETTER

to a former student (KAREN)
(with minor modifications)

Hi Karen,

My pursuit of a good picture of what was going on with the development of non-Euclidean geometry is starting to pull together. I can't resist passing on this summary to you.

1) Ever since the Greeks the mathematicians have wanted to replace the parallel axiom with either a simpler, more obvious axiom (such as the existence of similar non congruent triangles, or the existence of a rectangle), or a proof from the other axioms.

2) Saccheri published a book in 1733 (the year he died) called "Eulid Freed from every Flaw" which was so elegant that it quickly spread into the major university libraries, including Goettingen. Whereas previous attempts to clean up the parallel axiom had revolved around finding a substitute, he decided to derive a contradiction from assuming it failed.

   His basic starting point was to analyze what is now called a Saccheri quadrilateral using two perpendiculars of equal length erected on a line segment. He showed the two angles at the top of this quadrilateral were equal, and that either these angles in all such quads are right angles, or they are obtuse angles, or they are acute angles. The obtuse angle case gets knocked out very quickly by the exterior angle theorem: an exterior angle of a triangle exceeds either opposite interior angle.

   But the acute angle case proved stubborn. He proceeds to study parallel lines in this case (and perpendiculars to at least one of them) culminating in the claim that there are closest parallels (that I will call asymptotic parallels.

   After this brilliant analysis he goes off the deep end to get a contradiction. Actually he gives several short proofs that the acute angle hypothesis is not possible, all depending on either the behavior of points at infinity or analysis of infinitesimal segments. These proofs are all regarded as incorrect.

3) Lambert 1766 (published posthumously 1786 by one of the

Bernoulli's). He also has three cases, but in terms of the sum the angles of a triangle: $= 180$, > 180, and < 180. The acute angle case of Saccheri is the last, and he also focuses at length on it. He claims to prove that the area of a triangle ABC is proportional to the defect $180 - (A+B+C)$ in the sum of the angles. He only gives an example of his proof saying that if triangle T1 has half the area of triangle T2 then one can cut T2 into alot of small triangles that cover T1 twice. Then he uses the easy fact that the defect in a triangle T is equal to the sum of the defects of the triangles in a triangulation of T to show that T2 has twice the defect of T1. I do not see any way to perform his cutting up of T2 and repacking it twice into T1.

He says that in the obtuse case the area of a triangle is proportional to the excess $(A+B+C) - 180^\circ$, just as in ordinary spherical trigonometry where the area is R^2 times the excess, R being the radius of the sphere. This suggests to him that one might realize the acute angle geometry on a sphere of imaginary radius iR (since $(iR)^2 = -R^2$, turning the excess into a defect). But he does not pursue this.

4) Kaestner, a teacher of Bolyai and Gauss in Goettingen (in the 1790s), wrote a book on foundations of arithmetic and geometry back in 1757. In the preface he says that originally he thought a proof of the parallel postulate due to Hausens was correct, but upon learning from a friend that it had a flaw he proceeded to search for a correct proof, collecting in the process most of the extant literature on the subject of parallels, accumulating, as he said, "a small library" on the topic. [When he died his total library had 7,000 items --- I don't know how many were devoted to parallels.] He says in the preface that he had not found anything satisfactory in all of his searches regarding parallels.

   In 1763 his student Kluegel wrote a dissertation that gave, for the first time, a history of the study of parallels. (He examined some 30 incorrect proofs of the parallel axiom.) By the time Bolyai [1796] and Gauss [1795] arrived in Goettingen old prof Kaestner had given up hope of a proof of the parallel axiom being found.

   So it is no surprise that Bolyai and Gauss left Goettingen with the parallel axiom very much on their minds.

   Aside:
   [Gauss didn't have a very high opinon of Kaestner. He said that he was considered "the best mathematician among poets, and the

best poet among mathematicians". Evidently Bolyai senior kept a drawing Gauss had made as a student showing Kaestner standing at the blackboard writing out the steps to multiply two 3 digit numbers, making a mistake. (Gauss could correctly multiply two such numbers in his head almost instantly!)]

5) [The Gauss correspondence that I translated:
   www.thoralf.uwaterloo.ca/htdocs/geometry]

6) Taurinus publishes a book in 1826 in which he follows through on Gauss's comments to his uncle Schweikart that he had learned 4 or 5 years earlier. He notes that if the acute hypothesis holds then the non-Euclidean geometry is determined once a certain constant is known. He decides that since there is no reason for the constant to have any preferred value then it made no sense for our geometry to be non-Euclidean. But, nonetheless, he decides to pursue the implications of the acute angle hypothesis further, saying that it had occurred to him some time ago that since the obtuse angle hypothesis was like that of spherical geometry, the acute angle hypothesis should be like that of the opposite of spherical geometry. Then he proceeds to substitute imaginary arguments into the formulas of spherical trigonometry to formally derive trigonometry, as well as perimeter and area formulas for circles, in the acute angle geometry.

   These would essentially agree with the formulas that would appear in the later works of Bolyai and Lobaschewski.

7) Bolyai's Appendix to his father's book "An Attempt...." [to present a clear development of mathematics from first principles] focuses early on asymptotic parallels in the acute angle case. Suppose we are in such a geometry of space called G. He introduces the concept of a surface (now called a horosphere) that is normal to a pencil of asymptotic parallels.

   Let H be such a surface. (One can think of it somewhat as a sphere centered at a point at infinity where the pencil converges.) Define H-lines on H to be the intersection of H with G-planes normal to H. Bolyai then says that H with these H-lines satisfies all of Euclid's axioms for the plane. After noting that H-circles are also G-circles (with different centers), he proceeds to develop trigonometry and circumference formulas by going back and forth between G and H, using the fact that in the Euclidean plane H he knows what the trigonometry and circumference are. Bolyai's arguments, like those of the previous items, are largely based on the congruent triangles theorems (SAS, SSS) and an occasional use of continuous motion of a line.

There were a few places where I found Bolyai's arguments needed more consideration, but most of all they needed a model. He had no explicit description of his lines (say in R^3), so whether or not it held together seemed questionable. Also I did not like his treatment of area. For the circle he simply says: d/dr (Area) = Circumference, and then integrates his formula for the Circumference. I have not been able to come up with a workable definition of area in the acute angle geometry. "Squares" don't have right angles in the corners, so you can't even cut a "rectangle" up into squares. One could use the defn that the area of A is less than the area of B if you can cut A into a finite number of parts that fit disjointly inside B. But that seems very difficult to work with, although Eves book on geometry does show how to cut two triangles with the same defect into finitely many pairwise congruent triangles. [The notion of equal by finite decomposition is credited to Bolyai senior. After showing (in Euclidean geometry ?) that two planar polygons with the same area are equal by finite decomposition he turned to finding a proof for the case of two polyhedra with the same volume. This occupied a great deal of Bolyai junior's time. Later Max Dehn would give a counterexample.]

8) I found a brief but enlightening reference in:

John Stillwell, Mathematics and its History, Springer, 1989.

He says Beltrami was the first to construct a model of the Bolyai-Lobachewski non-Euclidean geometry of space [1868]:

Take the space to be {(x,y,z) : z > 0} with the metric

ds = (Euclidean metric) / z

The lines are semicircles perpendicular to z=0 as well as ordinary vertical half-lines; the planes are hemispheres perpendicular to z=0 as well as ordinary vertical half-planes. The horospheres are both the spheres in the upper half of R^3 tangent to the plane z = 0 as well as the constant Euclidean planes z = c > 0.

Stillwell doesn't discuss the notion of angle nor the issue of Euclid's first 28 propositions holding, including the results on congruent triangles that I think are needed to show that the reasoning of Bolyai can be considered correct. Presumably there are lots of isometries in Beltrami's model so you can move angles and triangles around.

Another item that bothered me was:
Bolyai seems to be claiming that the circumference of a circle in a (Euclidean) horosphere is the same whether considered as a circle in

the (Euclidean) horosphere or in the non-Euclidean space; or at least that the ratio of the circumferences of two circles in a (Euclidean) horosphere is the same when considered in the non-Euclidean space. This seems to need justification. However having the metric probably makes that clear....

Now I come to a philosophical/didactic issue: we teach things like "you can't duplicate the cube", or "you can't construct an 11-gon with straight-edge and compass" as though these things were completely settled. But what if our world is really one of the non-Euclidean worlds, say of Bolyai and Lobachewski, for example. Wouldn't it be more appropriate to say that the above questions of the Greeks for the idealistic Euclidean geometry were settled in the 19th century, but the more interesting questions of what constructions in our non-Euclidean geometry are possible lack a definitive solution. Johann Bolyai claims a proof that one can square the circle in the acute angle case in his Appendix. On the other hand, I can't even figure out how to trisect a line segment in Bolyai's geometry! In the acute angle geometry we don't have the rectangular Cartesian coordinates, no telling what the equation of a line looks like, or how one would develop a calculus tied to tangents and area.

Will our acceptance of Euclidean geometry eventually be regarded as a historical curiosity, something of use to the applied mathematicians as an approximation to the real world. I wonder about the solution to the equation:

$$\frac{\text{Aristotelian Logic}}{\text{Modern Logic}} = \frac{\text{Euclidean Geometry}}{\text{?????}}$$

Stan