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Downward Löwenheim-Skolem theorem

In 1915 Löwenheim gave a nearly complete proof of the fact that if a first-order formula φ has a model, then it has a finite or countable model. A tidy, correct version of this was given by Skolem in 1922. We will give the more general version for arbitrary sets of formulas and arbitrary languages.[1]

DEFINITION 1 Given a language $\mathcal{L}$ and a set of variables X let $\kappa = \max(\aleph_0, |\mathcal{L}|, |X|)$. κ is called the *power* of the language $\mathcal{L}$.

THEOREM 2 [Downward Löwenheim-Skolem] If Σ is a set of $\mathcal{L}$-formulas which is satisfiable, and the power of the language $\mathcal{L}$ is κ, then Σ can be satisfied by a model $(\mathbf{S}, \lambda)$ where $|\mathrm{S}| \leq \kappa$.

Proof. Let $(\mathbf{S}, \lambda) \models \Sigma$, and let Σ' be a Skolemization of Σ. Then expand $(\mathbf{S}, \lambda)$ to a model $(\mathbf{S}', \lambda)$ of Σ'. Let S'' be the subset of S generated by the constants and the range of λ. S'' is closed under the operations of $\mathbf{S}'$; so let $\mathbf{S}''$ be the structure obtained by restricting the functions and relations of $\mathbf{S}'$ to S''. One sees that $(\mathbf{S}'', \lambda) \models \Sigma'$, and that $|\mathrm{S}''| \leq \kappa$. By reducing back to the original language we have the desired model of Σ. ■

A remarkable consequence of this result is the existence of small models of axiomatic first-order set theory (provided any model exists).

COROLLARY 3 [Skolem] If ZF (or ZFC or vNBG) set theory has a model, it must have a countable model.

This seems to pose an obvious problem, because in such set theories one can prove that the power set of a set has strictly greater cardinality than the set — but if we have a countable model then all of our sets are countable. This is called Skolem's paradox.

[1]Mal'cev is credited for extending first-order languages to arbitrary cardinalities.

Exercises

Problem 1 Given an arbitrary group **G** show that there is a countable subgroup **H** such that Th(**H**) = Th(**G**).

References

[1] L. Löwenheim, Über Möglichkeiten im Relativkalkül. Math. Ann. **68** (1915), 169–207. [translated in *From Frege to Gödel*, van Heijenoort, Harvard Univ. Press, 1971.]

[2] Th. Skolem, Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit und Beweisbarkeit mathematischen Sätze nebst einem Theoreme über dichte Mengen. Videnskabsakademiet i Kristiania, Skrifter I, No. 4, 1919, 1–36. Also in "Selected Works in Logic by Th. Skolem", ed. by Jens Erik Fenstak, Scand. Univ. Books, Universitetsforlaget, Oslo, 1970, pp. 103–136. [The first section is translated in: *From Frege to Gödel*, van Heijenoort, Harvard Univ. Press, 1971, 252–263.]

[3] Th. Skolem, Einige Bemerkung zur axiomatischen Begrundung der Mengenlehre. Proc. 5^{th} Scand. Math. Congr. Helsinki, 1922, 217–232. [translation in *From Frege to Gödel*, van Heijenoort, Harvard Univ. Press, 1971.]