The Laws of Boole's Thought

Stanley Burris

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Abstract

The algebra of logic developed by Boole was not Boolean algebra. In this article we give a natural framework that allows one to easily reconstruct his algebra and see the difficulties it created for his successors.

It is well known that modern Boolean algebra is connected with the work of George Boole [1815–1864], namely with his two books on a mathematical treatment of logic:

- The Mathematical Analysis of Logic, 1847
- The Laws of Thought, 1854.

These two books will be referred to as Analysis and Laws. What is not well known is just how far removed Boole's work was from modern Boolean algebra, both in substance and in spirit. Contrary to popular belief Boole did not work with a two-element Boolean algebra, nor with the Boolean algebra of subsets of a given set. Boole was simply not doing Boolean algebra, nor Boolean rings. As a matter of fact, for more than a century no one really knew why Boole's algebra of logic worked. In 1869 Jevons said ([6], §7):

> The quasimathematical methods of Dr. Boole especially are so magical and abstruse, that they appear to pass beyond the comprehension and criticism of most other writers, and are calmly ignored.

Jevons was one of the rare writers on logic to admit that Boole's system was not built on principles that he could understand. But even Jevons, like so many others, had come to the conclusion that indeed Boole's system worked ([7], §175):

> It is not to be denied that Boole's system is consistent and perfect within itself. It is, perhaps, one of the most marvellous and admirable pieces of reasoning ever put together. Indeed, if ... the chief excellence of a system is in being *reasoned* and consistent within itself, then Professor Boole's is nearly or quite the most perfect system ever struck out by a single writer.

Boole wanted a symbolic algebra for sets[1] A, B, etc. He was fully aware of our favorite concepts of union, intersection, and complement (as well as symmetric difference).[2] But rather than adopting our modern approach of assigning symbols to these operations and then determining the laws that they satisfy, that is, the idempotent, commutative, etc., laws, Boole approached the task from a

[1] Actually Boole followed the logicians of his time and used the word *class*. But for a modern audience the word *set* seems preferable.

[2] See Laws, p. 56, to find Boole's mode of expressing the *union* and *symmetric difference*.

totally different point of view. He wanted to use the existing "common algebra", henceforth called *high school algebra*. Thus the laws of his algebra of logic were, for the most part, already given by what one learns in high school. We claim that the guiding principle of Boole's development was to *keep the algebra of logic as close as possible to high school algebra*. This is such a simple thesis, and yet it does not seem to have been expressed before. With this in mind it becomes much easier to follow the work of Boole.

The purpose of this note is to smooth out the development of Boole's algebra so that the reader will readily see how Boole was led to his peculiar use of partial operations, and understand why his algebra of logic was so mystifying to his successors.

Boole does use one of the fundamental operations on sets to establish the link with high school algebra, namely he lets the *intersection* of A and B be expressed by the *product* AB.[3]

From this one easily has the law[4]

$$A^2 \;=\; A.$$

Of course $A^2 = A$ is not a law of high school algebra, but Boole emphasizes that this is the only addition he has to make to obtain his algebra of logic.

Boole frequently writes $A^2 = A$ as $A(1 - A) = 0$. In high school algebra these are equivalent assertions. However there are other simple facts from high school algebra, such as '$A^2 = A$ implies $A = 0$ or $A = 1$', that he definitely wants to avoid. In spite of the fact that he said (Analysis, p. 18) that "all the processes of common algebra are applicable" to his algebra of logic, in reality one discovers that he has limited himself to pure equational deductions, that is, to deductions of the form

$$s_1 = t_1,\ \ldots,\ s_k = t_k \qquad \text{implies} \qquad s = t.$$

Thus, for example, '$2A = 0$ implies $A = 0$' is acceptable, but not '$AB = 0$ implies $A = 0$ or $B = 0$'.

Boole notes that one can express the universal affirmative categorical statement 'All A is B' by the equation $AB = A$. In modern notation this is equivalent to $A \subseteq B$. We will make use of this to show that Boole's choice of product to represent intersection, along with his desire to use high school algebra, will dictate the meanings of $0, 1, +, -$. To understand $\div$ will require his expansion theorem.

First let us consider which numbers can be interpreted as sets. If B is such a number then we must have $B^2 = B$, and thus B can only be 0 or 1. In the following keep in mind that we assume the equational laws and derivations of high school algebra are valid.

[3]Just how did Boole come to adopt this particular correspondence? This link is absolutely crucial to the eventual justification of Boole's algebra of logic. Hailperin [4] says that Boole's choice of the product notation for intersection was a consequence of the use of **selection operators** in his original (1847) algebra of logic. Boole had already achieved fame for his work with differential operators, and in 1847 he said that given a class A he would use the symbol a to denote the corresponding selection operator, defined in modern terms by $a(X) = A \cap X$. The composition of two such operators was expressed as their product, i.e., one has $(ab)(X) = a(b(X))$. Boole's 1847 system then started off by explaining why the equations $ab = ba$ and $aa = a$ hold. Later, in 1854, Boole switched from using selection operators to using classes. Here one finds his clearly stated use of the multiplicative notation AB for the intersection $A \cap B$.

Although Hailperin's reasoning is quite compelling, it is interesting to note that in 1847 De Morgan used, in his book Formal Logic, the product notation AB for the intersection, evidently without the motivation of operator notation. (He used A, B for the union. Unfortunately he did not have a notation for '='.)

[4]Actually Boole writes this law as $x^2 = x$. In this form a modern reader might be tempted to think that one can apply substitution and derive, for example, $(x + y)^2 = x + y$. But this is definitely not the case, so we take the liberty of using capital latin letters $A, B, \ldots$ as names for sets, to suggest that substitution is not automatically permitted. The symbols A, B, etc., that we use in the equations of Boole's algebra are to be thought of as *constants*, and not as universally quantified *variables*. Thus by 'the law $A^2 = A$' we really mean $A^2 = A$, $B^2 = B$, etc., for all the symbols A, B, etc., that one is using to name sets.

- **How can 0 be interpreted as a set?**

  To preserve $A0 = 0$ the only possible interpretation of 0 is the *empty set* since a set 0 must intersect every set in 0. Let us assume this interpretation holds.

- **How can 1 be interpreted as a set?**

  To preserve $A1 = A$ the only possible interpretation of 1 is the *universe* since a set 1 must intersect every set A in A. Let us assume this interpretation holds.

Next we consider the operations $+$ and $-$:

- **How can $A + B$ be interpreted as a set?**

  We are assuming that A and B are sets. For $A+B$ to be a set we need $A+B$ to be idempotent, that is, $(A+B)^2 = A+B$. This leads to $2AB = 0$, as $(A+B)^2 = A^2+2AB+B^2 = A+2AB+B$, and thus to $AB = 0$.[5] So a *necessary condition* is that A and B be disjoint.

  Now suppose that $AB = 0$. What set could $A + B$ be? To determine this let $C = A \cup B$, and assume $A + B$ is a set. Then from $A \subseteq C$ and $B \subseteq C$ we have $AC = A$ and $BC = B$. Adding these two equations together (and using some high school algebra) gives $(A + B)C = A + B$. But then $A + B \subseteq C$. Furthermore

  $$A(A + B) \;=\; AA + AB \;=\; A + 0 \;=\; A,$$

  so $A \subseteq A + B$. Likewise $B \subseteq A + B$. This leads to $C \subseteq A + B$, and thus $C = A + B$. So the only possible choice of a set for $A + B$ when A and B are disjoint is the set $A \cup B$. Let us assume this interpretation holds.

- **How can $A - B$ be interpreted as a set?** If $A - B$ is a set then by the idempotent property we have $(A - B)^2 = A - B$, and thus $2AB = 2B$, so $AB = B$. So a *necessary condition* is that $B \subseteq A$.

  Now suppose that $B \subseteq A$. What set could $A - B$ be? To determine this let $C = A \setminus B$, the set of elements in A but not in B. Then $BC = 0$, so by our previous convention we see that $B + C = B \cup C$. But $B \cup C = A$, so $B + C = A$, and thus $A - B = C = A \setminus B$. So the only possible choice of a set for $A - B$ when $B \subseteq A$ is the set difference $A \setminus B$. Let us assume this interpretation holds.

- It follows that $1 - A$ denotes the **complement** of A.

The results of our development so far agree perfectly with the system of Boole. However only in the case of the interpretation of 0 and 1 did he spell out, as we have, the reasons for his interpretation.[6] He simply defines $A + B$ when A and B are disjoint as the union, and eventually says that he can see no way to interpret $A + B$ otherwise,[7] but the details in our discussion above are not given. Likewise for $A - B$, Boole simply defines this as the set difference when $B \subseteq A$.

[5] Boole's use of expressions like $2AB$ have long been a source of irritation for readers of his work. Boole does not define 2,3, etc, but simply uses them as one would in high school algebra. Perhaps most important for us are the facts that $1 + 1 = 2$ and $s + s = 2s$.

[6] [Laws, p. 46] An important part of the following inquiry will consist in proving that the symbols 0 and 1 occupy a place, and are susceptible of an interpretation, among the symbols of Logic; ...

[7] [Laws, p. 66] The expression $x + y$ seems indeed uninterpretable, unless it be assumed that the things represented by x and the things represented by y are entirely separate; that they embrace no individuals in common.

Boole's first goal in 1847 was to show that he could handle the traditional Aristotelian logic with his algebraic system. His translations of the famous Aristotelian categorical propositions were as follows:

All A is B	$AB = A$	or	$A(1-B) = 0$
No A is B	$AB = 0$		
Some A is B	$V = AB$		
Some A is not B	$V = A(1-B)$.		

Notice that to handle the two 'particular' propositions he introduced a new set symbol V. If we recall that Aristotelian logic requires that all named classes be nonempty, then this approach is not unreasonable.

After discussing the possible *conversions* of propositions, such as the conversion of 'No A is B' into 'No B is A', he turned to the three line arguments known as *syllogisms*. His goal was to demonstrate that the conclusion in a valid syllogism was really the result of applying a simple high school algebra consequence of the solvability of two linear equations in one unknown.

First Boole translated the premises of an Aristotelian syllogism about three classes S, M, and P, where only S and P appear in the conclusion, as above. Then he used high school algebra to rewrite these in the form

$$\begin{aligned} aM + b &= 0 \\ cM + d &= 0, \end{aligned}$$

where the middle term M does not appear in any of coefficients a, b, c, d. Then he said that the most general conclusion obtained from eliminating M was given by the well known

$$ad - bc = 0.$$

(Unfortunately there were exceptional cases of syllogisms that needed special treatment.)

Let us work through one of his examples (see [1], p. 34), the 1st Figure AAA syllogism, by his method. In the following we give the usual form of the premises, then their translation into equations, and then further transformations using high school algebra to arrive at two linear equations in M:

All M is P	$M(1-P) = 0$	$(1-P)M = 0$
All S is M	$S(1-M) = 0$	$SM - S = 0$

On the right we can read off the coefficients: $a = 1-P$, $b = 0$, $c = S$, $d = -S$. Then the conclusion $ad - bc = 0$ must be $(1-P)(-S) - 0S = 0$, or applying high school algebra, $S(1-P) = 0$, which translates back into 'All S is P'.

Already, in this example, we run into one of the great stumbling blocks of Boole's work, at least for his successors. Note that in the second column above, the terms $M(1-P)$ and $S(1-M)$ in the equations always denote sets when the symbols S, M, P are interpreted as specific sets, as does the term $(1-P)M$ in the third column. However in the third column we see the term $SM - S$ for which this is not true. However if we choose sets S, M, P that make the premises true, then this term is also interpretable. But later we see the term $(-S)$, and unfortunately this will only be interpretable if $S = 0$, a condition not required by the premises.

In this example the terms in the premises are always interpretable, and the final form of the conclusion, $S(1-P) = 0$, is always interpretable. However there is an intermediate step with a term that need not be interpretable, even when the symbols S, M, P are assigned sets that make the premises true. This was the stumbling block.

There is a simple example of this problem in Boole's 1854 book (Laws, p. 123). From the premise $A = B$ Boole derives $A - B = 0$; then by squaring and using the idempotence of A and B he obtains $A - 2AB + B = 0$, which can be rewritten as the conclusion $A(1 - B) + B(1 - A) = 0$. Note that the terms in the premise and conclusion are always interpretable as sets when A,B are interpreted as sets. However, given an interpretation of the symbols A,B that makes the premise $A = B$ true, one encounters the subterm $2AB$ in an intermediate step. $2AB$ cannot denote a set unless it is idempotent, and this leads to $2AB = 0$, and hence $AB = 0$. Thus $2AB$ need not be interpretable for interpretations of A,B for which the premise holds.

Suppose the symbols in an argument have been assigned a fixed interpretation as sets. The remarkable fact about Boole's system is that if the terms in the premises are interpretable,[8] and those in a conclusion derived by Boole's methods are interpretable, then the argument '*premises* ∴ *conclusion*' is a correct argument about sets, even if the intermediate steps have terms that are not interpretable. This was a most puzzling situation for Boole's successors.

Boole does not discuss the possibility that a term used in one of his arguments might not be interpretable in his 1847 book, but later he tries to come to grips with the issue. Boole argued strenuously in his 1854 book that the use of uninterpretable terms in the intermediate steps of arguments in his system was sound because it was part of the *symbolical method*.[9] He gave an analogy with the use of $\sqrt{-1}$, which had no meaning (that he knew of), but was perfectly admissible in "the intermediate processes of trigonometry" (Laws, p. 69).

After completing the analysis of syllogisms Boole realized that there was no need to restrict his methods to syllogisms—he could accomodate any number of premises involving any number of sets. As an organizational tool in this more general setting he quickly hit upon the use of terms called *constituents*—in modern terminology we could say that linear combinations $\sum m_i t_i$ of constituents (over the integers) provide normal forms for the expressions in Boole's algebra of logic.

The constituents for a given finite list $A_1, \ldots, A_n$ of symbols are the 2^n products t_j whose factors are the A_i or their complements $1 - A_i$. Thus for the symbols A, B we have the four constitutents

$$AB \quad (1 - A)B \quad A(1 - B) \quad (1 - A)(1 - B)$$

Note that each constituent t denotes an intersection of sets and is thus interpretable. An integer multiple mt of t will be interpretable precisely when m is 0 or 1.

If $t_1, \ldots, t_m$ are the constituents on a given finite set of symbols then Boole observes that:

$$\begin{aligned} t_i^2 &= t_i && \text{for all } i \\ t_i t_j &= 0 && \text{if } i \neq j \\ t_1 + \cdots + t_m &= 1 \end{aligned}$$

Boole presented his **Expansion Theorem** in 1847. It allowed one to expand a term as a linear

[8] Boole did not give a definition of what it meant for a term to be interpretable, given an interpretation of the symbols $A, B, \ldots$—he merely gives a couple of simple examples. What is not clear is when he meant for a term like $A(A - B)$ to be considered as interpretable. Clearly it is interpretable if $B \subseteq A$, but otherwise $A - B$ is not interpretable. However the term $A(A - B)$ equals, in Boole's system, the term $A - AB$, which is always interpretable. We leave it to the reader's discretion as to how to resolve such cases. For our discussion it is only important that there exist some terms that are interpretable, and some that are not.

[9] [Laws, p. 68] There exist, in fact, certain general principles relating to the use of symbolical methods, which, as pertaining to the particular subject of Logic, I shall first state ...

2nd, That the formal processes of solution or demonstration be conducted throughout in obedience to all the laws determined as above, without regard to the question of the interpretability of the particular results obtained.

combination of constituents:

$$\begin{aligned} f(A) &= f(1)A + f(0)(1-A) \\ f(A,B) &= f(1,1)AB + f(1,0)A(1-B) + f(0,1)(1-A)B + f(0,0)(1-A)(1-B) \\ &\textit{etc.} \end{aligned}$$

The reader can easily see that these hold for terms built up from $+, -, \cdot$, and symbols A, B, etc., by using the idempotence of the symbols, and by noting each of the symbols is expressible as a sum of constituents (those that mention the symbol, and not its complement). Substituting these sums for the symbols leads to an expression for f that is a linear combination of the constituents:

$$f(A, \ldots) \;=\; c_1 t_1 + \;\cdots\; + c_n t_n$$

The coefficients c_i do not depend on the choice of idempotent elements A, etc. As 0 and 1 are idempotents, the last equation holds when we substitute 0 or 1 for each of the symbols A, etc, and this gives the expansion theorem.

Once we have the expansion theorem it is clear that any equation $f = 0$ is equivalent to a system of equations $t_i = 0$ where each t_i is a constituent. Let us call such equations **constituent equations**. Since constituent equations are meaningful assertions about sets, Boole can claim that any equation $f = 0$ can be interpreted, even though the expression f may not be interpretable, i.e., f may not denote a set. For example, $A + B$ may not be interpretable as a set in Boole's system, but as the expansion of $A + B$ is $2AB + A(1 - B) + (1 - A)B$ we see that the equation $A + B = 0$ is equivalent to the system $AB = A(1 - B) = (1 - A)B = 0$, which is equivalent in modern terminology to $A \cup B = 0$.

Suppose we are given an interpretation of the symbols $A, B, \ldots$. With the expansion theorem we can see that a term $f(A, B, \ldots)$ is equal to some interpretable $g(A, B, \ldots)$ if and only if $f(A, B, \ldots)$ is idempotent. Boole only states that idempotence is necessary for $f(A, B, \ldots)$ to be interpretable, and he refers to idempotence as "the condition of interpretability" (Laws, p. 93).

Boole says (Laws, p. 92) that a term $f(A, B, \ldots)$ is *independently interpretable* if, for every interpretation of the symbols $A, B, \ldots$ as sets the term is interpretable. One readily sees that $f(A, B, \ldots)$ is equal to some independently interpretable $g(A, B, \ldots)$ if and only if all the coefficients in the expansion of f are either 0 or 1.

In Laws Boole introduced a new fundamental principle that he said was the cornerstone of his work. It is a beautiful principle, but he offered no justification for its correctness. It says basically that an (equational) argument

$$f_1 = 0, \;\ldots\;, \; f_k = 0 \qquad \text{implies} \qquad f = 0$$

in his algebra of logic is correct if, when one substitutes 0s and 1s for the symbols A, etc, in any manner whatever, the argument is correct in the ordinary numbers. We will call this Boole's **Rule of 0 and 1**.[10]

This principle can be justified from what we have developed so far, namely by replacing each of the equations by a system of constituent equations. This principle has been mistakenly taken as evidence that Boole worked with a two-element algebra. The fact that the symbols A, B, etc., take on the values 0 or 1 does not mean that the terms $A + B$, etc., also take on such values. If A and

[10] [Laws, pp. 37-38] Let us conceive, then, of an Algebra in which the symbols $x, y, z, \&c.$ admit indifferently of the values 0 and 1, and of these values alone. The laws, the axioms, and the processes, of such an Algebra will be identical in their whole extent with the laws, the axioms, and the processes of an Algebra of Logic. Difference of interpretation will alone divide them. Upon this principle the method of the following work is established.

B take on the value 1 then $A+B$ takes on the value 2 in the ordinary number system. Thus if we want to use Boole's rule to check the validity of the commutative law $A+B=B+A$ we could write up the cases in truth-table style:

A	B	$A+B$	$B+A$
1	1	2	2
1	0	1	1
0	1	1	1
0	0	0	0

As the last two columns agree in each of the rows we have verified the law. Rather than a two-element Boolean algebra, Boole used the ordinary **integers** as his basic reference algebra!

Boole's last, and main, theorem was his **Elimination Theorem**[11], which said that to determine the most general conclusion $g(B_1, \ldots, B_n) = 0$ involving only the symbols $B_1, \ldots, B_n$ from the single premise $f(A_1, \ldots, A_m; B_1, \ldots, B_n) = 0$ one merely had to substitute 0s and 1s for the A_i in all possible ways, and then take the product. Thus for $f(A_1, A_2, B_1) = 0$, eliminating the A_i leads to the most general conclusion $f(1,1,B_1) \cdot f(1,0,B_1) \cdot f(0,1,B_1) \cdot f(0,0,B_1) = 0$. Boole could have easily proved the elimination theorem from his Rule of 0 and 1 mentioned above, but he preferred to try more direct methods.

In order to apply the elimination theorem when one is given several premises $f_1 = 0, \ldots, f_n = 0$, Boole gave methods to combine these into an equivalent single premise. One of the methods was to use $f_1^2 + \cdots + f_n^2 = 0$.

Boole used the **division operation** essentially in the following situation. Given an equation $f(A, B, \ldots) = 0$, where f involves only the operations $+, -, \cdot$, find an expression for A in terms of the other symbols. His first step was to factor out the A in f to obtain the form $g(B, \ldots)A + h(B, \ldots) = 0$. Then he formally put

$$A \;=\; -\frac{h(B, \ldots)}{g(B, \ldots)}$$

and declared that the meaning of the right hand side is to be obtained through the expansion theorem. Thus for example one has, from $AB = C$, the formal division

$$A = \frac{C}{B},$$

and then the expansion theorem yields

$$\begin{aligned} A &= \frac{1}{1}BC + \frac{1}{0}(1-B)C + \frac{0}{1}B(1-C) + \frac{0}{0}(1-B)(1-C) \\ &= BC + \frac{1}{0}(1-B)C + \frac{0}{0}(1-B)(1-C) \end{aligned}$$

Boole's system requires that one must set $(1-B)C = 0$ as a side condition, to counter the coefficient $1/0$, and remove that term from the expression for A. The fraction $0/0$ he said is indeterminate, and should be replaced with a new set symbol V, yielding

$$A \;=\; BC + V(1-B)(1-C).$$

[11] The idea of having an elimination theorem was further developed by Schröder and then Skolem to give the modern concept of *elimination of quantifiers*.

The side condition says that $C \subseteq B$, which of course we would need if $AB = C$. And the solution for A says that A is the set BC [which is C] union some portion of $(1 - B)(1 - C)$ [which is $1 - B$]. Boole's treatment of division by the expansion method does indeed lead to correct results by modern standards.

Boole's successors were troubled by several parts of his algebra of logic. First the idea to tie the algebra of logic to the ordinary algebra of numbers seemed fantastic. Boole said that it was probably beyond our ability to comprehend just why the laws of thought should be so closely related to the laws of number.[12]

Second his system was built on partially defined operations, and yet Boole's equational logic (of high school algebra) proceeded as if he were working with totally defined functions. His justification of this liberty was shallow, namely that such things were allowed in the symbolic method.

Third his operation of division seemed mysterious, an operation whose meaning was derived from the expansion theorem, and depended on working with coefficients like 1/0 and 0/0.

Fourth his use of equations to express *some* was regarded by C.S. Peirce, and subsequently Schröder, as an error. Schröder would adopt the negated equation $AB \neq 0$ to express "Some A is B".

Jevons referred to Boole's work as using "dark and symbolic processes" ([7], §174.). He decided to develop the algebra of logic without any reference to numbers and without any attempt to assume á priori that high school algebra applied. He kept the notation of sum and product, but defined the sum to be the union. Then, in modern fashion, Jevons does a fairly complete job of determining the laws and rules of inference of the equational logic of modern Boolean algebra (with constants A, B, etc, but no variables). Before this appeared in print he corresponded with Boole (see [8]), in 1863, about the obscure side of Boole's work, and said that the self-evident law $A + A = A$ should be added. Boole would only agree that $A + A = 0$ implies $A = 0$, but not that $A + A = A$, and eventually broke off the correspondence because they could not get past this point. In his last letter to Boole Jevons takes a more contrite position, saying that although Boole's system "probably is perfectly consistent with itself", one still needed to consider which of the systems would more accurately "correspond to the Logic of common thought". But Boole never responded. Within a year Boole had died, and so far as we know, Boole was never ready to endorse the Boolean algebra of union, intersection and complement as a valid approach to the algebra of logic.

In the nineteenth century the study of symbolic systems for sets was referred to as the Algebra of Logic. Boole introduced this name, and it continued to be used until 1913. In the 1880s C.S. Peirce referred to 'Boolian algebra', but by that he meant whatever Boole had done (and he did not particularly like it). Schröder, a follower of Peirce's work,[13] wrote up an excellent treatment of the subject in 1890 in the first of his three volumes titled *Algebra der Logik*.[14]

[12] [Laws, p. 11] Whence is it that the ultimate laws of Logic are mathematical in their form; why they are, except in a single point [namely the law $A^2 = A$], identical with the general laws of Number; and why in that particular point they differ;—are questions upon which it might not be very remote from presumption to endeavour to pronounce a positive judgement. Probably they lie beyond the reach of our limited faculties.

[13] The ideas of Jevons were partly rediscovered by Peirce. In 1880 Peirce gave a development of the algebra of logic, as pertains to sets, that took the partial order properties of $\subseteq$ as the starting point. (This is the reverse of Jevons position, who claimed Aristotle had made a big mistake by choosing 'All A is B' to be more fundamental than 'A = B'.) The operations of $+$ and $\times$ were introduced as the glb and lub in this setting. Before introducing the elements 0,1, and the complement, Peirce proved the basic equational laws for $+$ and $\times$. We call these the laws of **lattices**. Unfortunately he claimed that the distributive law followed easily.

Peirce expanded the algebra of logic to include the algebra of binary relations (inspired by earlier work of De Morgan on binary relations), and quantifiers.

[14] Schröder followed Peirce in starting with the partial ordering as the fundamental concept. The fragment of the algebra of logic involving just $+, \times$ he called the *calculus of groups* [Gruppenkalkül]. Schröder was using the word 'group' in the sense of a set closed under some kind of operations. So this was a calculus of closed sets. And from

By following Peirce's development Schröder parted with Jevon's purely equational approach. And whereas Jevons was able to pull together an essentially complete proof system (axioms and rules of inference) for his equational approach, the rules of inference to handle the Peirce-Schröder approach were never organized into a coherent proof system.

In Schröder's treatment we see the familiar operations of union, intersection, and complement applied to the collection of subsets of a given set. In 1904 Huntington [5] wrote a paper giving several axiom systems for the algebra of logic (for sets). It was in this paper that he considered arbitrary models of the axioms, an important step for the modern subject of Boolean algebra. In 1913, in a follow-up to Huntington's paper, Sheffer [11] referred to the models of any such system of axioms for the algebra of logic as **Boolean algebras**. He introduced an axiom system (based on the now-famous Sheffer stroke) that provided "A set of five independent postulates for Boolean algebras". This is how Boole's name became attached to the modern subject of Boolean algebra.

It is unfortunate for the historically minded reader that Boole's algebra has been so vaguely, and often incorrectly, treated in the literature. For more than a century after Boole writers on the algebra of logic, and on Boolean algebra, failed to describe why Boole's system worked, and often misinterpreted portions of his work. At times they would attack pieces of his work without being able to find any real fault in what he had done. After all, his methods gave correct answers. Schröder's monumental volumes completely avoid discussing the details of the development of Boole's algebra. The results of Boole that were recognized as correct in the context of the operations of union, intersection, and complement were acknowledged by Schröder, and proved in this new context, e.g., the expansion and elimination theorems. But there was no critical evaluation of Boole's algebra.

Following Schröder's carefully written treatment of the algebra of logic for sets, the interest in finding out why Boole's system worked just seemed to fade away. It was not until 1976 that one finds a successful analysis. Theodore Hailperin [3], using basic knowledge of modern ring theory, shows how one can use the rings $\mathbf{Z}^U$ to provide models[15] for the system of Boole, and finally we can see why it is that Boole's system works. For an overview of Hailperin's analysis of Boole's system we can strongly recommend his expository paper [4].

References

[1] George Boole, **The Mathematical Analysis of Logic,** Being an Essay Towards a Calculus of Deductive Reasoning. Originally published in Cambridge by Macmillan, Barclay, & Macmillan, 1847. Reprinted in Oxford by Basil Blackwell, 1951.

[2] George Boole, **An Investigation of The Laws of Thought** on Which are Founded the Mathematical Theories of Logic and Probabilities. Originally published by Macmillan, London, 1854. Reprint by Dover, 1958.

the modern point of view it was the study of lattices. Schröder had found counterexamples to Peirce's claim that the distributive law held for this fragment of the algebra of logic—they appear in appendices to the first volume.

The full algebra of logic for sets, including 0,1, and complement, he called the *calculus of identity* [Identischekalkül]. Later Skolem, and subsequently Hilbert and Ackermann, would prefer the name *calculus of classes* [Klassenkalkül]. However the name 'algebra of logic' continued to be used by some authors, such as Huntington, for just the set portion of the algebra of logic.

[15] Hailperin [4] says that the most important models are the subrings of the $\mathbf{Z}^U$ generated by the idempotent elements—such subrings give algebras of **signed multisets**.

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[10] Ernst Schröder, **Algebra der Logik, Vols. I–III**. 1890–1910, Chelsea reprint 1966.

[11] Henry Maurice Sheffer, *A set of five independent postulates for Boolean algebras, with application to logical constants.* Trans. Amer. Math. Soc. **14** (1913), 481–488.