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A NOTE ON VARIETIES OF UNARY ALGEBRAS

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If G and G^* are two non-isomorphic (congruence) simple finite groups, then they generate distinct varieties (see [6], p. 166).

B. Jónsson has proved in [5] that the same result is valid for lattices. The following construction will show that this property fails to hold for unary algebras (for terminology see [3]).

Let $\langle G, \cdot\rangle$ be a multiplicative group and consider the left translation algebra $\mathfrak{A}(G) = \langle G, \mathfrak{F}\rangle$, where $\mathfrak{F} = \{f_a \colon a \in G, f_a(x) = ax\}$.

LEMMA 1. *A basis for the identities of* $\mathfrak{A}(G)$ *is given by* $\{f_a f_b = f_{ab} \colon a, b \in G\}$.

Proof. An immediate consequence of the observation that $f_{a_1} \cdots f_{a_n} = f_{b_1} \cdots f_{b_m}$ iff $a_1 \ldots a_n = b_1 \ldots b_m$.

An equivalence relation θ on G is *left compatible* if $\langle x, y\rangle \in \theta \Rightarrow \langle zx, zy\rangle \in \theta$ for all z in G (see [1]). It is easy to see that θ is left compatible iff $\theta\,[1]$ is a subgroup of G and the equivalence classes are precisely the left cosets of $\theta\,[1]$.

LEMMA 2. *θ is a congruence for $\mathfrak{A}(G)$ iff θ is left compatible with G.*

Proof. Straightforward.

For convenience of notation, if H is a subgroup of G, let $\mathfrak{A}(G)/H$ denote the algebra with the carrier G/H and with $f_a(bH) = abH$ $(b \in G)$ as fundamental operations (G/H denotes the set of left cosets). Also, define $N(G, H) = \bigcap \{\lambda H \lambda^{-1} \colon \lambda \in G\}$.

THEOREM. *A basis for the laws of $\mathfrak{A}(G)/H$ (where $H \neq G$) is given by*

$$\{f_a f_b = f_{ab} \colon a, b \in G\} \cup \{f_a = f_1 \colon a \in N(G, H)\}.$$

Proof. In view of Lemma 1 we only need to determine the a, b in G such that $f_a = f_b$ in $\mathfrak{A}(G)/H$; but this is equivalent to $f_{b^{-1}a} = f_1$. If $f_a = f_1$ in $\mathfrak{A}(G)/H$, then $f_a(\lambda H) = \lambda H$ for all $\lambda \in G$, i.e. $a \in \bigcap \{\lambda H \lambda^{-1} \colon \lambda \in G\} = N(G, H)$, and conversely.

EXAMPLE. Let G be the alternating group on 5 elements, and let H and K be two maximal subgroups of different orders. Then $\mathfrak{A}(G)/H$

and $\mathfrak{A}(G)/K$ are simple and non-isomorphic, and since $N(G, H) = N(G, K) = \{1\}$, it follows from the Theorem that they generate the same variety.

PROBLEM 1. Does there exist a variety of semi-groups generated by each of two non-isomorphic simple finite semi-groups? (**P 704**).

Following a suggestion of Djokovic the author was able to conclude the existence of any given finite number of non-isomorphic simple finite unary algebras which generate the same variety (1) by examining the maximal subgroups of $PSL(2, 2^f)$ for suitable f (cf. [4], p. 213). However, it is easy to show that we cannot increase this to an infinite number for the following reasons. Let $\mathscr{V}$ be a variety of unary algebras generated by a finite algebra. Since congruence simple (and cardinality greater than two) implies at most one subalgebra, it follows that every congruence simple algebra in $\mathscr{V}$ would be a homomorphic image of the free algebra on one generator, or a two element algebra, and thus there could only be a finite number of congruence simple algebras in $\mathscr{V}$.

On the other hand, Comer has exhibited in [2] a variety of semi-groups which can be generated by any one of an infinite number of non-isomorphic subdirectly irreducible finite semi-groups.

PROBLEM 2. Does there exist an infinite number of non-isomorphic simple finite algebras which generate the same variety? (**P 705**).

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Added in proof. T. Karnofsky (Berkeley) has announced positive result for the two problems; see Notices of the American Mathematical Society 17 (1970), p. 939.

(1) This generalizes Wille [7].

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