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REPRESENTATION THEOREMS FOR CLOSURE SPACES

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The purpose of this paper is to describe necessary and sufficient conditions on a closure C so that there is an abstract algebra such that the natural closure associated with the abstract algebra is precisely C, where the family of maps in the algebra are required to satisfy some special conditions. A discussion of this problem when the family of maps can be arbitrary is given in [1].

In the following let the set S be fixed. If C is a map from 2^S to 2^S and n is a positive integer, then C^n will denote the n-fold composition of C with itself; and for P a subset of S, let $\overline{\overline{P}}$ be the cardinality of P.

A mapping C from 2^S to 2^S is a *pre-closure* if for any P and Q the following two conditions hold:

C.1. $P \subset C(P)$.

C.2. $P \subset Q \Rightarrow C(P) \subset C(Q)$.

A pre-closure C which satisfies

C.3. $C^2(P) \subset C(P)$ for all P contained in S

is a *closure*.

A pre-closure (or closure) which satisfies the compactness condition

C.4. For any $P \subset S$ and for any $x \in C(P)$ there is a finite Q contained in P such that $x \in C(Q)$

will be called *algebraic*.

The following propositions will give the basic properties of pre-closures which will be needed. The proofs of (i) and (ii) are trivial and that of (iii) follows by an easy induction argument.

(i) *If C is a pre-closure, then C^n is a pre-closure for any positive integer n.*

(ii) *If C is a pre-closure, then the map $\bigcup C^n$ which takes a subset P into $\bigcup_{n\geqslant 1} C^n(P)$ is a pre-closure.*

(iii) *If C is an algebraic pre-closure, then C^n is an algebraic pre-closure for $n = 1, 2, \ldots$*

(iv) *If C is an algebraic pre-closure, then $\bigcup C^n$ is an algebraic closure.*

Proof. Properties C.1 and C.2 for $\bigcup C^n$ follow by (ii). Let $x \in \bigcup\limits_{n\geqslant 1} C^n(P)$. Then there is an n such that $x \in C^n(P)$, and by (iii) there is a finite Q contained in P such that $x \in C^n(Q)$. Therefore $x \in \bigcup\limits_{n\geqslant 1} C^n(Q)$ and C.4 is established for $\bigcup C^n$.

To show property C.3, let

$$x \in \bigcup_{m\geqslant 1} C^m\Big[\bigcup_{n\geqslant 1} C^n(P)\Big].$$

Since C.4 holds for $\bigcup C^m$, there is a finite Q contained in $\bigcup\limits_{n\geqslant 1} C^n(P)$ such that $x \in \bigcup\limits_{m\geqslant 1} C^m(Q)$. The sets $C^n(P)$, $n = 1, 2, \ldots$, form a nest, and since Q is finite, there is a k such that $Q \subset C^k(P)$. Therefore

$$\bigcup_{m\geqslant 1} C^m(Q) \subset \bigcup_{m\geqslant 1} C^m\big(C^k(P)\big) = \bigcup_{m\geqslant 1} C^{m+k}(P) \subset \bigcup_{m\geqslant 1} C^m(P).$$

Thus $x \in \bigcup\limits_{m\geqslant 1} C^m(P)$. Therefore

$$\bigcup_{m\geqslant 1} C^m\Big[\bigcup_{n\geqslant 1} C^n(P)\Big] \subset \bigcup_{m\geqslant 1} C^m(P),$$

and we have C.3.

Let C_1 and C_2 be pre-closures. Define $C_1 \subset C_2$ if for all P, $C_1(P) \subset C_2(P)$. An easy induction leads to

(v) $C_1 \subset C_2 \Rightarrow C_1^k \subset C_2^k$ for $k = 1, 2, \ldots$

Hence,

(vi) $C_1 \subset C_2 \Rightarrow \bigcup C_1^n \subset \bigcup C_2^n$.

So far we have constructed pre-closures and closures from given pre-closures. Now to go in the other direction, let C be a closure and N a positive integer. Then define C_N by

$$C_N(P) = \bigcup\{C(Q): Q \subset P \text{ and } \overline{\overline{Q}} \leqslant N\}.$$

Then

(vii) for C a closure, C_N is an algebraic pre-closure for any positive integer N.

A closure C will be called *N-ary* if N is a positive integer such that $C = \bigcup C_N^n$ [1]. It is easily argued that if C is N-ary, then it is also $(N+k)$-ary for k a non-negative integer. We also see that an N-ary closure is necessarily algebraic.

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[1] I would like to express my indebtedness to E. Marczewski for suggesting that N-ary closures (and later N-ary algebras) be included in this paper.

Next will follow two theorems which give rather satisfying connections between N-ary closures and algebraic pre-closures.

THEOREM 1. *Let C be an N-ary closure and suppose C^* is an algebraic pre-closure such that $C^* \subset C$. Then $C = \bigcup C^{*n}$ if and only if*

$$C(Q) = \bigcup_{n \geqslant 1} C^{*n}(Q)$$

for all Q such that $\overline{\overline{Q}} \leqslant N$.

Proof. First we note that $C^* \subset C$, hence $\bigcup C^{*n} \subset C$, since C is a closure. The proof in one direction is trivial, so we assume $C(Q) = \bigcup C^{*n}(Q)$ for all Q such that $\overline{\overline{Q}} \leqslant N$. Then, for any P,

$$\begin{aligned} C_N(P) &= \bigcup\{C(Q): Q \subset P, \overline{\overline{Q}} \leqslant N\} \\ &= \bigcup\{C^{*n}(Q): Q \subset P, \overline{\overline{Q}} \leqslant N\} \\ &\subset \bigcup_{n \geqslant 1} C^{*n}(P). \end{aligned}$$

Thus we have $C_N \subset \bigcup C^{*n}$, and since $\bigcup C^{*n}$ is a closure by (iv), we can apply (vi) to obtain $\bigcup C_N^n \subset \bigcup C^{*n}$. Since C is N-ary, the conclusion follows.

THEOREM 2. *Let C be a closure. Then C is N-ary if and only if there exists a pre-closure C^* such that*

T2.1. $C = \bigcup C^{*n}$;

T2.2. *for any P, $C^*(P) = \bigcup \{C^*(Q): Q \subset P, \overline{\overline{Q}} \leqslant N\}$.*

Proof. If C is an N-ary closure, then we can simply choose C^* to be C_N. For the converse assume we have a C^* such that T2.1 and T2.2 are satisfied. Since $C = \bigcup C^{*n}$, then $C^* \subset C$. From T2.2, for any P,

$$C^*(P) = \bigcup \{C^*(Q): Q \subset P, \overline{\overline{Q}} \leqslant N\} \subset \bigcup \{C(Q): Q \subset P, \overline{\overline{Q}} \leqslant N\} = C_N(P).$$

Thus $C^* \subset C_N$, and from (vi) and T2.1 we have $C \subset \bigcup C_N^n$. Now, since C is a closure and $C_N \subset C$ for any positive integer N, we can apply (vi) to obtain $\bigcup C_N^n \subset C$, and the theorem is proved.

If C is an N-ary closure and in addition $C(Q)$ is countable (possibly finite) for all Q such that $\overline{\overline{Q}} \leqslant N$, then C will be called *operational*, and N will be called an *index* for C. If we restrict our attention to operational closures, then we can state a result parallel to Theorem 2, but considerably stronger. The next theorem will show that every operational closure can be expressed in terms of a pre-closure whose properties will be the key to the final theorem.

THEOREM 3. *Let C be a closure. Then C is operational if and only if there exist a pre-closure C^* and positive integers M and L such that*

T3.1. $C = \bigcup C^{*n}$;

T3.2. *for any* P, $C^*(P) = \bigcup \{C^*(Q): Q \subset P, \overline{\overline{Q}} \leqslant M\}$;

T3.3. $\overline{\overline{C^*(Q)}} \leqslant L$ *for all* Q *such that* $\overline{\overline{Q}} \leqslant M$.

Proof. Suppose we have a C^* such that the three conditions are satisfied. Then from T3.1, T3.2 and Theorem 2 it follows that C is M-ary. From T3.2 and T3.3 it follows that $C^*(P)$ is finite when P is finite, and then from T3.1 we see that $C(P)$ is countable when P is finite. Thus C is operational with index M.

The proof of the converse is a little more involved. Let C be operational (with index N). If Q is any subset of S such that $\overline{\overline{Q}} \leqslant N$, then we know that $C(Q)$ is countable and therefore we can assume that the elements of $C(Q)$ have been indexed by positive integers. Then we can write $C(Q) = \{a_1^Q, a_2^Q, \ldots\}$ (where the sequence might be finite). Furthermore require that the above sequences satisfy: 1) all members of a given sequence are distinct; 2) the members of Q appear first in $a_1^Q, a_2^Q, \ldots$

Now for any subset R of S such that $\overline{\overline{R}} \leqslant N+1$ define $C^*(R)$ to be the set

$$R \cup \{a_1^Q\} \cup \{a_{n+1}^Q: Q \subset R, \overline{\overline{Q}} \leqslant N, \text{ and } a_n^Q \in R\}$$

with the understanding that the expression $\{a_1^{\emptyset}\}$ is to be deleted from the above if $C(\emptyset)$ is empty.

The following three properties of C^* are easy consequences of the definition of C^*, where, of course, R has no more than $N+1$ elements:

C^*.1. $R \subset C^*(R)$;

C^*.2. $C^*(R) \subset C(R)$;

C^*.3. $C^*(R) = \bigcup \{C^*(Q): Q \subset R\}$.

Because of C^*.3 it is possible to extend C^* to all of 2^S by

C^*.4. $C^*(P) = \bigcup \{C^*(R): R \subset P, \overline{\overline{R}} \leqslant N+1\}$.

We will assume C^* to be so extended, and then note that from the four properties above it easily follows that C^* actually is an algebraic pre-closure. Then from C^*.2, C^*.4 and the fact that C.2 is satisfied by C we can conclude $C^*(P) \subset C(P)$ for any P, i.e., $C^* \subset C$.

Let Q be such that $\overline{\overline{Q}} \leqslant N$. If $C(Q) = \emptyset$, then necessarily $C^*(Q) = C(Q)$. For this special case it is immediate that

$$C(Q) = \bigcup_{n \geqslant 1} C^{*n}(Q).$$

So now we will assume that $C(Q) \neq \emptyset$. Then from the restriction 2 on the sequence a_i^Q and from the definition of $C^*(Q)$ we have $a_1^Q \in C^*(Q)$. By a simple induction argument it follows that $a_n^Q \in C^{*n}(Q)$ for n a positive integer, and thus $C(Q)$ is contained in $\bigcup_{n\geqslant 1} C^{*n}(Q)$. Combining this result with that of the previous paragraph gives $C(Q) = \bigcup_{n\geqslant 1} C^{*n}(Q)$ for all Q such that $\overline{\overline{Q}} \leqslant N$, and therefore we have from Theorem 1 that $C = \bigcup C^{*n}$.

Now if we let $M = N+1$ and $L = N+2+2^{N+1}(N+1)$, then it is straightforward to show that the three conditions of Theorem 3 are satisfied by C^*.

Before proceeding we need additional notation and definitions. A mapping f from a finite Cartesian product of S into S will be called *finitary*. If F is a family of finitary maps, then (S, F) will be called an *algebra* (or *abstract algebra*). An algebra (S, F) is N*-ary* if every operation belonging to F is at most N-ary. The mapping C defined by $C(P) =$ *the smallest subset of S containing P and closed under the elements of F* is an algebraic closure (see [1]). The closure so defined is called the *closure induced by* (S, F). Conversely, given a closure C, then any algebra whose induced closure is C will be called an *algebraic representation of C* (or *of the closure space* (S, C)). The following representation theorem is indeed pleasing (²):

THEOREM 4. *A closure is induced by an N-ary algebra iff it is N-ary.*

Proof. Suppose C is an N-ary closure. Then fix an ordering in S and for each a in S define

$$f_a(x_1, \ldots, x_N) = \begin{cases} a \quad \text{if} \quad a \in C(\{x_1, \ldots, x_N\}); \\ \text{the first element in this order among } x_1, \ldots, x_N \text{ if} \\ \hfill a \notin C(\{x_1, \ldots, x_N\}). \end{cases}$$

and then define $F = \{f_a : a \in S\}$. (S, F) is clearly an N-ary algebra, and one readily verifies that C is induced by F. Furthermore, we note that in the above construction the operations have been chosen *symmetrical*.

Now suppose (S, F) is an N-ary algebra. For each P contained in S define

$$C^*(P) = \bigcup \{f(P^n) : f \in F, \ \text{domain}\,(f) = S^n\} \cup P.$$

We can easily show that C^* is an algebraic pre-closure and that $\bigcup C^{*n}$ is the closure induced by (S, F). Then Theorem 2 gives the desired conclusion.

In the introduction it was mentioned that if we allow F to be infinite, then the problem of algebraic representation is well known — in fact

(²) Due to E. Marczewski.

every algebraic closure has an algebraic representation. The following theorem will cover the case where F is restricted to be finite ([3]):

THEOREM 5. *Let C be an algebraic closure. Then C has an algebraic representation by some (S, F) with F finite if and only if C is operational.*

Proof. Assume C has an algebraic representation by (S, F) with F finite. Let M be the maximum n such that there is an $f \in F$ and f maps S^n into S. Then for all P contained in S define

$$C^*(P) = \bigcup\{f(P^n): f \in F,\ \text{domain } (f) = S^n\} \cup P.$$

If $\overline{\overline{Q}} \leqslant M$, then $\overline{\overline{C^*(Q)}} \leqslant (\overline{\overline{F}})(M^M)+M$. Let the latter expression be L. As in the proof of Theorem 4 we can claim $C = \bigcup C^{*n}$. Properties T3.2 and T3.3 follow from the definitions of C^*, M and L, so by Theorem 3 we know that C is operational.

For the converse assume that C is operational with index N. Then let C^*, M and L satisfy the conditions of Theorem 3. For all Q such that $\overline{\overline{Q}} \leqslant M$ let $a_1^Q, \ldots, a_L^Q$ be some fixed ordering of the elements of $C^*(Q)$ where 1) if necessary some element of $C^*(Q)$ can appear more than once so that the sequence has L members, and 2) all the elements of $C^*(Q)$ appear in the sequence. Then, define the finitary functions f_i from S^M to S for $1 \leqslant i \leqslant L$ by: $f_i(x_1, \ldots, x_M) = a_i^Q$ where Q is the set of elements $\{x_1, \ldots, x_M\}$. Again we note that the operations have been chosen symmetric.

Now, using property T3.2 it follows rather readily from the definition of the f_i that

$$C^*(P) = \bigcup\{f_i(P^M): 1 \leqslant i \leqslant L\},$$

and then because of property T3.1 we can verify that C is induced by (S, F), where of course $F = \{f_i: 1 \leqslant i \leqslant L\}$, and therefore C has an algebraic representation with F finite.

Remark. From the proof of Theorem 5 we see that an operational closure with index N has a representation by an algebra (S, F), where F has at most $N+2+2^{N+1}(N+1)$ maps, and each of the maps are $(N+1)$-ary.

The following example will show that Theorem 5 provides a simple necessary and sufficient condition for an important class of closures to have an algebraic representation by an algebra (S, F) with F finite. Let S be a lattice, and for P a subset of S, let $C(P)$ be the filter generated by P (i.e. the least filter containing P). Then C has an algebraic representation by an algebra with a finite number of maps *iff* for each $x \in S$, $C(\{x\})$

([3]) I would like to thank Allen S. Davis for suggesting this problem in [2].

is countable. To show this we need only note that $C = \bigcup C_2^n$, and for $x, y \in S$ it is true that $C_2(\{x, y\})$ is equal to $C_2(\{\inf(x, y)\})$. From the remark following Theorem 5 we see that if we have such a representation, then F need consist of at most 27 ternary maps.

REFERENCES

[1] P. M. Cohn, *Universal algebra*, New York 1965.

[2] Allen S. Davis, *Closure spaces*, Mimeographed notes, University of Oklahoma, 1966.

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